SIAM J. MATH. ANAL.
Vol. 27, No. 3, pp. 835–864, May 1996

ON SOME SHARP REGULARITY ESTIMATIONS OF L^2-SCALING FUNCTIONS*

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Abstract. Let f be a compactly supported L^2-solution of the two-scale dilation equation and α be the L^2-Lipschitz exponent of f. We prove, in addition to other results, that there exists an integer $k \geq 0$ such that (i) $\frac{1}{h^{2\alpha}|\ln h|^k}\int_{-\infty}^{\infty}|f(x+h)-f(x)|^2dx \approx p(h)$ as $h \to 0^+$, where p is a nonzero bounded continuous function with $p(2h) = p(h)$, and (ii) for $s > \alpha$, there exists a nonzero bounded continuous q (depends on s) with $q(2T) = q(T)$ and $\frac{1}{T^{2(s-\alpha)}(\ln T)^k}\int_{-T}^{T}|\omega^s\hat{f}(\omega)|^2d\omega \approx q(T)$ as $T \to \infty$. The above α and k can be calculated through a transition matrix. These improve the previous result of Cohen and Daubechies concerning the Besov space containing f and Villemoes's result on the Sobolev exponent of $\hat{f}$.

Key words. asymptotics, compactly supported L^2-solutions, dilation equation, Fourier transformation, Lipschitz exponent, spectral radius, regularity, Tauberian theorem, wavelet

AMS subject classifications. 26A15, 26A18, 39A10, 42A05

1. Introduction.

The existence, regularity, and orthogonality of the compactly supported L^2-solution (notation: L^2_c-solution) of the two-scale dilation equation

$$f(x) = \sum_{n=0}^{N} c_n f(2x-n) \tag{1.1}$$

have been studied in great detail (e.g., [CD], [CH], [D], [DL1], [DL2], [E], [H], [LW1], [V], [W]). In much of the literature, the techniques and emphases are on the frequency domain, i.e., the consideration of the Fourier transformation of (1.1),

$$\hat{f}(\omega) = m_0\left(\frac{\omega}{2}\right)\hat{f}\left(\frac{\omega}{2}\right),$$

where $m_0(\omega) = \frac{1}{2}\sum c_n e^{in\omega}$ and $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$. On the other hand, there are linear algebraic methods on the time domain which also yield many important results concerning continuous solutions ([DL1], [DL2], [CH], [W]) and L^p-solutions [LW1].

In this paper, we continue our study through the second method. For the L^2-case, the existence and regularity results in [CD] and [V] are largely derived from the $(2N-1)\times(2N-1)$ matrix $\mathbf{W}_N$ associated with the operator $\mathbf{A}$ on functions in the frequency domain defined by

$$\mathbf{A}g(\omega) = \left|m_0\left(\frac{\omega}{2}\right)\right|^2 g\left(\frac{\omega}{2}\right) + \left|m_0\left(\frac{\omega}{2}+\pi\right)\right|^2 g\left(\frac{\omega}{2}+\pi\right)$$

(which was introduced in [CR]). The matrix $\mathbf{W}_N$ actually comes out more naturally in the time-domain consideration. For $g \in L^2(\mathbb{R})$ supported in $[0, N]$, if we let $\mathbf{a}(g)$ denote the autocorrelation vector of $a_n(g) = \int g(x+n)\overline{g(x)}dx$, $|n| < N$, and $Sg(x) = \sum_{n=0}^{N} c_n g(2x-n)$, then

$$\mathbf{a}(Sg) = \frac{1}{2}\mathbf{W}_N\mathbf{a}(g) \tag{1.2}$$

*Received by the editors February 4, 1994; accepted for publication (in revised form) August 4, 1994.

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(Proposition 3.1). This is the most basic and important relationship in the L^2-consideration. Note that if g is an L^2_c-solution of (1.1), then $Sg = g$, and it follows that $\mathbf{a}(g)$ is a 2-eigenvector of $\mathbf{W}_N$. Villemoes [V] essentially proved that (1.1) has an L^2_c-solution if and only if $\mathbf{W}_N$ has a 2-eigenvector which is positive definite. Here we will give another characterization of the existence of the L^2_c-solution based on $\mathbf{W}_N$ and two other associated matrices $\mathbf{T}_0$ and $\mathbf{T}_1$ used in [DL1], [DL2], [CH], [W], and [LW1]. We also simplify a theorem of Cohen and Daubechies [CD, Thm. 4.3] concerning the eigenvalues of $\mathbf{W}_N$ and the Riesz basis property.

Our main objective is to consider the regularity of the L^2_c-solutions. Assuming $\sum c_n = 2$, let

$$\Lambda_{\max} = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathbf{W}_N^+ \text{ and } |\lambda| \neq 2\}$$

($\mathbf{W}_N^+$ is certain truncation of $\mathbf{W}_N$ to the positive coordinates) and let

$$\alpha = -\ln(\Lambda_{\max}/2)/(2\ln 2);$$

then $0 < \alpha \leq 1$. In [V], Villemoes proved that if f is an L^2_c-solution of (1.1) and if $r < \alpha$, then $\int_{-\infty}^{\infty} |\omega^r \hat{f}(\omega)|^2 d\omega < \infty$ so that f is in the Sobolev space $H^r(\mathbb{R})$ for $r < \alpha$, and the Sobolev exponent of f is α. By using the Littlewood–Paley method, Cohen and Daubechies [CD] showed that f is in the Besov space $B_2^{r,\infty}$ for all $r < \alpha$ (an equivalent definition of Besov space $B_2^{r,\infty}$ is $\sup_{h>0} \frac{1}{h^r}\|\Delta_h f\|_2 < \infty$, where $\Delta_h f = f(\cdot + h) - f(\cdot)$ [P]). They left out the critical case when the exponent $r = \alpha$. Here we obtain some sharp estimations of the regularity of f and the decaying rate of $\hat{f}$, which improve the previous results.

THEOREM 1.1. *Let f be an L^2_c-solution of* (1.1). *Let m be the highest order among those eigenvalues λ of $\mathbf{W}_N^+$ such that $|\lambda| = \Lambda_{\max}$; then*

$$\frac{1}{h^{2\alpha}|\ln h|^{m-1}} \int_{-\infty}^{\infty} |\Delta_h f|^2 = p(h) + o(h)$$

as $h \to 0^+$, where p is a nonzero bounded continuous multiplicative periodic function of period 2 (i.e., $p(2h) = p(h), h > 0$). (The order of an eigenvalue λ is the power of the factor $(x - \lambda)$ in the minimal polynomial.)

If we define the L^2-Lipschitz exponent of $g \in L^2(\mathbb{R})$ by

$$L^2\text{-Lip}(g) = \inf\{s : 0 < \limsup_{h\to 0^+} \frac{1}{h^s}\|\Delta_h g\|_2\},$$

then it follows from Theorem 1.1 that the L^2-Lipschitz exponent of the L^2_c-solution f is α, which is also the Sobolev exponent of f for $0 < \alpha < 1$. To study higher-order regularity, the usual assumption is the l-sum rule, $l > 1$. Here we do not need such a hypothesis, we use the lth-order difference $\Delta_h^{(l)} f$ to define the L^2-Lipschitz order for $0 \leq \alpha \leq l$ ($\Lambda_{\max}$ has to be redefined). Theorem 1.1 can be extended accordingly with the exception that when α is an integer, then the logarithmic factor can be of order $m - 1$ or $m - 2$.

For the frequency domain, we have the following asymptotic result (including higher-order α).

THEOREM 1.2. *Under the above assumptions, for any $s > \alpha$*

$$\frac{1}{T^{2(s-\alpha)}(\ln T)^k} \int_{-T}^{T} |\omega^s \hat{f}(\omega)|^2 d\omega \approx q(T)$$

as $T \to \infty$, *where* q *is a nonzero bounded continuous function with* $q(2T) = q(T)$; *if* α *is* not *an integer, then* $k = m - 1$; *if* α *is an integer, then* $k = m - 1$ *or* $m - 2$.

Theorem 1.1 corresponds to Theorem 5.4 later in the text. The main idea of the proof is to extend the identity (1.2) to another autocorrelation vector $\mathbf{\Phi}(h) = [\Phi_0(h), \Phi_1(h), \cdots, \Phi_N(h)]$, where Φ_n is defined by

$$\Phi_n(h) = \int_{-\infty}^{\infty} \Delta_h f(x+n)\Delta_h f(x)dx,$$

and show that for any λ-eigenvector $\mathbf{u}$ of $\mathbf{W}_N^+, \lambda \neq 0, 2$, $\langle \mathbf{\Phi}(h), \mathbf{u}\rangle = h^{2\beta}p(h)$, where $\beta = -\ln(\lambda/2)/(2\ln 2)$ and p is a nonzero bounded continuous multiplicative periodic function (Lemma 5.1, Theorem 5.2). The most involved step is to show that $\langle \mathbf{\Phi}(h), \mathbf{u}\rangle \neq 0$ (Lemma 4.3), which makes use of a classical result of L. Schwartz on the *mean periodic functions* [Sch], [K], [RL]. Theorem 1.2 is contained in Theorem 5.7 and in §6, it is derived from Theorem 1.1 by using a new form of Tauberian theorem proved in [L3].

We remark that equation (1.1) actually describes a certain self-similarity of f. The self-similar measures in fractal theory are also defined by the same class of functional equation [Hu]. The genuine ideas of calculating the asymptotic properties in Theorems 1.1 and 1.2 are already contained in [L1], [LW2], [S1], [S2], and in particular in [L2].

The Daubechies four-coefficient scaling function $D_4 = f$ provides an interesting example for the above theorems (see §6 and the appendix). It follows from a direct calculation and Theorem 1.1 that $\Lambda_{\max} = \frac{1}{2}$, $\alpha = 1$, and the regularity is given by $\frac{1}{h^2|\ln h|}\int_{-\infty}^{\infty} |\Delta_h f|^2 \approx p(h)$ as $h \to 0^+$. It is also differentiable a.e. [D], [DL2], but the derivative is not in $L^2(\mathbb{R})$ in view of the asymptotic regularity behavior as $h \to 0^+$.

We organize the paper as follows. In §2, we introduce the transition matrix $\mathbf{W}_N$ as well as the two associated matrices $\mathbf{W}$ and $\mathbf{W}_N^+$. In §3, we consider some basic properties of the transition matrices in connection with the autocorrelation functions. For completeness, we simplify the existence characterization of the L_c^2-solutions proved in [LW1]. We also give a short proof of a theorem in [CD] concerning the eigenvalues of $\mathbf{W}_N$ when the solution has the Riesz basis property (Theorem 3.7). In §4 we set up the basic lemmas for the proof of Theorem 1.1, Lemma 4.3 being the most important one. Section 5 contains the proof of Theorems 1.1 and 1.2. Section 6 is concerned with the higher-order difference and the L^2-Lipschitz exponent $\alpha > 1$. At the end, we also include an appendix which contains some graphic implementations of the theorems where the functional equation (1.1) takes only four coefficients.

2. The transition matrices. For any sequence $\{c_n\} \in \ell^1(\mathbb{Z})$, we let

$$\omega_n = \sum_{k\in\mathbb{Z}} c_k c_{k-n}, \quad n \in \mathbb{Z}.$$

Then ω_n is the convolution of the two sequences $\{c_n\}$ and $\{c_{-n}\}$; $\{\omega_n\} \in \ell^1(\mathbb{Z})$ and $\omega_{-n} = \omega_n$. We define the infinite matrix $\mathbf{W}$ by

$$\mathbf{W} = [\omega_{i-2j}] = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & \omega_1 & \omega_{-1} & \omega_{-3} & \cdots \\ \cdots & \omega_2 & \omega_0 & \omega_{-2} & \cdots \\ \cdots & \omega_3 & \omega_1 & \omega_{-1} & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and $\mathbf{W}_N$ is the restriction of $\mathbf{W}$ on the entries $-N \le i, j \le N$. We also define

$$\mathbf{W}^+ = \begin{pmatrix} \omega_0 & \omega_{-2} & \omega_{-4} & \cdots \\ \omega_1 + \omega_{-1} & \omega_{-1} + \omega_{-3} & \omega_{-3} + \omega_{-5} & \cdots \\ \omega_2 + \omega_{-2} & \omega_0 + \omega_{-4} & \omega_{-2} + \omega_{-6} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

that is, each entry of $\mathbf{W}^+$ is given by

$$w_{ij}^+ = \begin{cases} \omega_{-2j} & \text{if } i = 0, \\ \omega_{i-2j} + \omega_{-i-2j} & \text{if } i > 0. \end{cases}$$

Geometrically, $\mathbf{W}^+$ is obtained by first deleting the left-half part of the columns of $\mathbf{W}$, then reflecting the upper half of this truncated matrix with respect to the zeroth row and adding it to the lower half. Similarly, we can truncate the matrix $\mathbf{W}_N$ to obtain $\mathbf{W}_N^+$.

When there is no confusion, we use $\mathbf{u}$ to denote the column vectors $[u_0, \dots, u_n]^t$, $[u_{-n}, \dots, u_0, \dots, u_n]^t$, and $[\dots, u_{-1}, u_0, u_1, \dots]^t$ ($[\cdot]^t$ denotes the transpose). We define $\mathbf{F} : \mathbb{C}^{N+1} \to \mathbb{C}^{2N+1}$ by

$$\mathbf{F}(\mathbf{u}) = \left[\frac{u_N}{2}, \dots, \frac{u_1}{2}, u_0, \frac{u_1}{2}, \dots, \frac{u_N}{2}\right]^t, \quad \mathbf{u} \in \mathbb{C}^{N+1},$$

and $\mathbf{G} : \mathbb{C}^{2N+1} \to \mathbb{C}^{N+1}$ by

$$\mathbf{G}(\mathbf{u}) = [u_0, u_1 + u_{-1}, \dots, u_N + u_{-N}]^t, \quad \mathbf{u} \in \mathbb{C}^{2N+1}.$$

It is clear that the adjoints of $\mathbf{F}$ and $\mathbf{G}$ are given by

$$\mathbf{F}^*(\mathbf{u}) = \left[u_0, \frac{1}{2}(u_1 + u_{-1}), \dots, \frac{1}{2}(u_N + u_{-N})\right]^t, \quad \mathbf{u} \in \mathbb{C}^{2N+1},$$

and

$$\mathbf{G}^*(\mathbf{u}) = [u_N, \dots, u_1, u_0, u_1, \dots, u_N]^t, \quad \mathbf{u} \in \mathbb{C}^{N+1}.$$

By a λ-eigenvector of a matrix $\mathbf{M}$, we mean a right eigenvector corresponding to the eigenvalue λ. The basic eigen properties of $\mathbf{W}_N$ and $\mathbf{W}_N^+$ are related as follows.

PROPOSITION 2.1. *If $\mathbf{u} \in \mathbb{C}^{N+1}$ is a λ-eigenvector of $\mathbf{W}_N^+$ ($(\mathbf{W}_N^+)^*$, resp.), then $\mathbf{F}(\mathbf{u})$ ($\mathbf{G}^*(\mathbf{u})$, resp.) $\in \mathbb{C}^{2N+1}$ is a λ-eigenvector of $\mathbf{W}_N$ ($(\mathbf{W}_N)^*$, resp.).*

Conversely, if $\mathbf{u} \in \mathbb{C}^{2N+1}$ is a λ-eigenvector of $\mathbf{W}_N$ ($(\mathbf{W}_N)^*$, resp.), then $\mathbf{G}(\mathbf{u})$ ($(\mathbf{F}^*)(\mathbf{u})$, resp.) is either 0 or a λ-eigenvector of $\mathbf{W}_N^+$ ($(\mathbf{W}_N^+)^*$, resp.).

Proof. Using elementary linear algebra and the fact that $\omega_n = \omega_{-n}$ for all $n \in \mathbb{Z}$, we have

$$\mathbf{W}_N \circ \mathbf{F} = \mathbf{F} \circ \mathbf{W}_N^+ \quad \text{and} \quad \mathbf{G} \circ \mathbf{W}_N = \mathbf{W}_N^+ \circ \mathbf{G}. \tag{2.1}$$

Suppose $\mathbf{u} \in \mathbb{C}^{N+1}$ is a λ-eigenvector of $\mathbf{W}_N^+$; then $\mathbf{F}(\mathbf{u}) \neq 0$ and by (2.1),

$$\mathbf{W}_N(\mathbf{F}(\mathbf{u})) = \mathbf{F}(\mathbf{W}_N^+\mathbf{u}) = \mathbf{F}(\lambda\mathbf{u}) = \lambda\mathbf{F}(\mathbf{u}).$$

On the other hand, suppose $\mathbf{u} \in \mathbb{C}^{2N+1}$ is a nonzero λ-eigenvector of $\mathbf{W}_N$; then

$$\mathbf{W}_N^+(\mathbf{G}(\mathbf{u})) = \mathbf{G}(\mathbf{W}_N\mathbf{u}) = \mathbf{G}(\lambda\mathbf{u}) = \lambda\mathbf{G}(\mathbf{u}).$$

The statements for the adjoints follow from the dual relationship of (2.1):

$$\mathbf{F}^* \circ (\mathbf{W}_N)^* = (\mathbf{W}_N^+)^* \circ \mathbf{F}^* \quad \text{and} \quad (\mathbf{W}_N)^* \circ \mathbf{G}^* = \mathbf{G}^* \circ (\mathbf{W}_N^+)^*. \tag{2.2}$$

Remark. If $\mathbf{u}$ is a λ-eigenvector of $\mathbf{W}_N$, then the proposition implies that $\mathbf{v} = \mathbf{F}(\mathbf{G}(\mathbf{u}))$ and $\mathbf{w} = \mathbf{u} - \mathbf{v}$ are also eigenvectors of $\mathbf{W}_N$ provided that they are not zero. Note that $\mathbf{v}$ is a symmetric and $\mathbf{w}$ is antisymmetric. If all the λ-eigenvectors of $\mathbf{W}_N$ are antisymmetric, then λ is not an eigenvalue of $\mathbf{W}_N^+$.

If $c_n = 0$ for all $n \in \mathbb{Z} \setminus \{0, 1, \ldots, N\}$, then $\omega_n = 0$ for all $|n| > N$, and

$$\mathbf{W}_N = \begin{pmatrix} \omega_N & \omega_{N-2} & \cdots & \omega_{-N+2} & \omega_{-N} & 0 & \cdots & 0 & 0 \\ 0 & \omega_{N-1} & \cdots & \omega_{-N+3} & \omega_{-N+1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \omega_1 & \omega_{-1} & \omega_{-3} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \omega_2 & \omega_0 & \omega_{-2} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \omega_3 & \omega_1 & \omega_{-1} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \omega_{N-1} & \omega_{N-3} & \cdots & \omega_{-N+1} & 0 \\ 0 & 0 & \cdots & 0 & \omega_N & \omega_{N-2} & \cdots & \omega_{-N+2} & \omega_{-N} \end{pmatrix}.$$

PROPOSITION 2.2. *Suppose $\sum_n c_{2n} = \sum_n c_{2n+1} = 1$ and $c_n = 0$ for all $n \in \mathbb{Z} \setminus \{0, 1, \ldots, N\}$. Then 2 is an eigenvalue of the matrices $\mathbf{W}_N$, $\mathbf{W}_{N-1}$, and $\mathbf{W}_N^+$, the vector $[1, \ldots, 1]^t \in \mathbb{C}^{2N+1}$ (or $\mathbb{C}^{2N-1}$) is a 2-eigenvector of $\mathbf{W}_N$ ($\mathbf{W}_{N-1}$, respectively), and $[1, 2, \ldots, 2]^t \in \mathbb{C}^{N+1}$ is a 2-eigenvector for $\mathbf{W}_N^+$.*

Proof. Note that the sum of each row of $\mathbf{W}_N$ is 2. Hence 2 is an eigenvalue; the corresponding eigenvector is $[1, 1, \ldots, 1]^t$ and Proposition 2.1 implies that 2 is also an eigenvalue of $\mathbf{W}_N^+$ with eigenvector $[1, 2, \ldots, 2]^t$.

3. The autocorrelation function. Let L_c^2 denote the set of all L^2-functions with compact supports. We call the solution of (1.1) a *scaling function*. It is well known that if $f \in L^1(\mathbb{R})$, then $\operatorname{supp} f \subseteq [0, N]$. For convenience, we assume that the c_n's are real, where $c_n = 0$ for all $n \in \mathbb{Z} \setminus \{0, 1, \ldots, N\}$, so that the solution is also real. Note that $\sum c_n = 2^m$, where $m \geq 1$ is a necessity condition for the existence of an L^1-solution f; if $m > 1$, then f is the $(m-1)$th derivative of another L^1-scaling function corresponding to the coefficients $\{2^{-(m-1)} c_n\}$ [DL1]. We will assume, without loss of generality, that $\sum c_n = 2$ throughout the paper.

For $g : \mathbb{R} \to \mathbb{R}$, we define

$$(Sg)(x) = \sum_{n=0}^{N} c_n g(2x - n).$$

It is easy to show that if $\operatorname{supp} g \subseteq [0, N]$, then $\operatorname{supp}(Sg) \subseteq [0, N]$ also. For each such g, we let

$$a_n(g) = \int_{-\infty}^{\infty} g(t+n) g(t)\, dt, \quad n \in \mathbb{Z},$$

be the nth autocorrelation of g defined on $\mathbb{Z}$. It is clear that $a_n(g) = a_{-n}(g)$ and $a_n(g) = 0$ for all $|n| \geq N$. By slightly abusing notations, we use $\mathbf{a}(g)$ to denote the autocorrelation vectors of g:

$$\mathbf{a}(g) = [\ldots, a_{-1}(g), a_0(g), a_1(g), \ldots]^t \quad \text{or} \quad [a_{-k}(g), \ldots, a_0(g), \ldots, a_k(g)]^t,$$

depending on the situation. Let $\mathbf{e}_n$ be the vector (finitely or infinitely many entries) with 1 on the nth entry and 0 otherwise. The major property of the transition matrix $\mathbf{W}$ defined in §2 is given in the following proposition.

PROPOSITION 3.1. *Let $g \in L^2(\mathbb{R})$ be supported in $[0, N]$. Then*

$$\mathbf{a}(Sg) = \frac{1}{2}\mathbf{W}^*\mathbf{a}(g), \tag{3.1}$$

where $\mathbf{W}^*$ is the adjoint of $\mathbf{W}$. In particular,

$$\int_{-\infty}^{\infty} |(Sg)(t)|^2 dt = \frac{1}{2}\langle \mathbf{a}(g), \mathbf{W}\mathbf{e}_0\rangle = \frac{1}{2}\langle \mathbf{a}(g), \mathbf{W}_{N-1}\,\mathbf{e}_0\rangle.$$

Proof. The proof is based on the following observation: for $n \in \mathbb{Z}$,

$$\begin{aligned}
a_n(Sg) &= \int_{-\infty}^{\infty} (Sg)(t+n)(Sg)(t)\,dt \\
&= \sum_{i,j\in\mathbb{Z}} c_j c_i \int_{-\infty}^{\infty} g(2t+2n-j)g(2t-i)\,dt \\
&= \frac{1}{2}\sum_{i,j\in\mathbb{Z}} c_j c_i \int_{-\infty}^{\infty} g(t+2n+i-j)g(t)\,dt \\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}} \left(\sum_{i\in\mathbb{Z}} c_i c_{i-(k-2n)}\right) a_k(g) \\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}} \omega_{k-2n} a_k(g) \\
&= \frac{1}{2}[\mathbf{W}^*\,\mathbf{a}(g)]_n
\end{aligned}$$

(the second equality holds since we assume that $c_n = 0$ for all $n \in \mathbb{Z} \setminus \{0, 1, \ldots, N\}$). The last identity in the proposition holds due to the fact that $a_n(g) = 0$ for all $|n| \geq N$.

Remark. For $y \in \mathbb{R}$, if we let $a_n^{(y)}(g) = \int_{-\infty}^{\infty} g(t+n-y)g(t)dt$, then the same calculation yields

$$\mathbf{a}^{(y)}(Sg) = \frac{1}{2}\mathbf{W}^*\mathbf{a}^{(2y)}(g). \tag{3.1$'$}$$

We will use this fact in Theorem 3.7.

Recall that a sequence $\{u_n\}_{n=-\infty}^{\infty}$ is called *positive definite* if for any finite sequence $\{\xi_n\}$, $\sum u_{m-n}\xi_m\bar{\xi}_n \geq 0$. It is well known that the autocorrelation sequence $\{a_n(g)\}$ (letting $a_n(g) = 0$ for all $|n| \geq N$) is positive definite.

PROPOSITION 3.2. *Suppose f is a nonzero L_c^2-solution of* (1.1); *then $\mathbf{a}(f)$ is a 2-eigenvector of $(\mathbf{W}_{N-1})^*$, $\sum a_n(f) \neq 0$, and $\{a_n(f)\}_{n=-\infty}^{\infty}$ is a positive-definite sequence.*

Proof. In view of Proposition 3.1 and the remark above, we need only show that $\sum a_n(f) \neq 0$. This follows from the well-known Poisson formula $\sum a_n(f)e^{in\omega} = \sum |\hat{f}(\omega + 2\pi n)|^2$ and the sum is strictly positive for $\omega = 0$.

The existence of a vector satisfying the above conditions also implies the existence of an L_c^2-solution of (1.1), which has been observed by Villemoes in [V] (where he uses $\sum a_n(f)e^{inx} \geq 0$ instead of using the fact that $\{a_n(f)\}_{n=-\infty}^{\infty}$ is positive definite).

In order to construct the L_c^2-solution f of the dilation equation (1.1), we can formally proceed as follows: take a function g with supp$g \subseteq [0, N]$ and consider $\{S^k(g)\}_{k=1}^{\infty}$. If this sequence converges in L^2 to a function f, then f will be a solution to (1.1). Equivalently, we can write

$$S^k(g) = g + \sum_{l=0}^{k-1} S^l(Sg - g) \tag{3.2}$$

and consider the convergence of the series $\sum_{l=0}^{k-1} S^l(Sg - g)$. Let

$$\mathbf{T}_0 = [c_{2i-j-1}]_{1 \leq i,j \leq N} = \begin{pmatrix} c_0 & 0 & 0 & \dots & 0 \\ c_2 & c_1 & c_0 & \dots & 0 \\ c_4 & c_3 & c_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \dots & c_{N-1} \end{pmatrix},$$

$$\mathbf{T}_1 = [c_{2i-j}]_{1 \leq i,j \leq N} = \begin{pmatrix} c_1 & c_0 & 0 & \dots & 0 \\ c_3 & c_2 & c_1 & \dots & 0 \\ c_5 & c_4 & c_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \dots & c_N \end{pmatrix}.$$

These matrices were used in [DL1], [DL2], [CH], and [W] to study the continuous scaling solutions. In [LW1, Thm. 4.3], the authors also use such matrices to give a necessary and sufficient condition for the existence of the L_c^p-solutions; for the L^2-case, the criterion is reduced to consider the eigenvalues of $\mathbf{W}$. The proof is simplified later in Theorem 3.4. First, we state a very useful lemma concerning $\mathbf{T}_0 + \mathbf{T}_1$ which is proven in [LW1].

LEMMA 3.3. *Suppose* $\sum_{n=0}^{N} c_n = 2$. *Then the following hold:*

(i) 2 *is an eigenvalue of* $\mathbf{T}_0 + \mathbf{T}_1$.

(ii) *If* $\mathbf{v}$ *is a* 2-*eigenvector of* $\mathbf{T}_0 + \mathbf{T}_1$, *let* $g = \sum_{n=0}^{N-1} v_n \chi_{[n-1,n)}$; *then*

$$\left[\int_0^1 S^k g, \dots, \int_{N-1}^{N} S^k g\right]^t = \mathbf{v}.$$

(iii) *For* $1 \leq p < \infty$, *let* $f \in L_c^p(\mathbb{R})$ *be the* L_c^p-*solution of* (1.1) *and let* $\mathbf{v} = [\int_0^1 f, \dots, \int_{N-1}^{N} f]^t$; *then* $\mathbf{v}$ *is a* 2-*eigenvector of* $\mathbf{T}_0 + \mathbf{T}_1$. *For such* $\mathbf{v}$, *if we let* g *be defined as in* (ii), *then* $\{S^k(g)\}$ *converges back to* f *in the* L^p-*norm.*

THEOREM 3.4. *Suppose* $\sum_{n=0}^{N} c_n = 2$. *Let* $\mathbf{v}$ *be a* 2-*eigenvector of* $\mathbf{T}_0 + \mathbf{T}_1$ *and let* $g = \sum_{n=0}^{N-1} v_n \chi_{[n,n+1)}$. *Let* $H_{\mathbf{v}}$ *be the smallest invariant subspace of* $\mathbf{W}_{N-1}$ *containing* *the autocorrelation vector* $\mathbf{a}(Sg-g)$. *Then* (1.1) *has a nonzero* L^2_c*-solution if and only if all the eigenvalues of* $\mathbf{W}_{N-1}$ *restricted to* $H_{\mathbf{v}}$ *have modulus less than* 2.

Proof. Let $\tilde{g} = Sg - g$. Note that by Proposition 3.1, we have

$$\begin{aligned}\|S^l\tilde{g}\|^2 &= \int_0^N |S^l\tilde{g}(t)|^2\,dt \\ &= \frac{1}{2^l}\langle \mathbf{a}(\tilde{g}),\ \mathbf{W}^l_{N-1}\mathbf{e}_0\rangle \\ &= \langle \frac{1}{2^l}(\mathbf{W}^l_{N-1})^*\,\mathbf{a}(\tilde{g}),\ \mathbf{e}_0\rangle.\end{aligned}$$

The assumption that $\frac{1}{2}(\mathbf{W}_{N-1})^*$ restricted on $H_{\mathbf{v}}$ has spectral radius less than 1 implies that $\{\frac{1}{2^l}(\mathbf{W}^l_{N-1})^*\mathbf{a}(\tilde{g})\}$ converges to zero geometrically as $l \longrightarrow \infty$; so does $\{\|S^l\tilde{g}\|^2\}$. Consequently, $S^k(g) = g + \sum_{l=0}^{k-1} S^l\tilde{g}$ converges in L^2. Let f be the limit. Then $f \in L^2_c(\mathbb{R})$; $f \neq 0$ because by Lemma 3.3(ii),

$$\left[\int_0^1 S^l\tilde{g}, \dots, \int_{N-1}^N S^l\tilde{g}\right] = 0$$

so that

$$\left[\int_0^1 f, \dots, \int_{N-1}^N f\right] = \left[\int_0^1 g, \dots, \int_{N-1}^N g\right] = \mathbf{v} \neq \mathbf{0}.$$

To prove the converse, we observe that (3.2) and Proposition 3.1 imply that

$$\frac{1}{2^l}(\mathbf{W}^l)^*\mathbf{a}(Sg-g) = \mathbf{a}(S^l(Sg-g)) \longrightarrow 0 \quad \text{as} \quad l \longrightarrow \infty.$$

It follows that all the eigenvalues of $\mathbf{W}_{N-1}$ restricted to $H_{\mathbf{v}}$ have modulus less than 2.

For the special case where $\sum_n c_{2n} = \sum_n c_{2n+1} = 1$, Theorem 3.4 yields a simple criterion for the existence of the L^2_c-solution (see also [CD, Thm. 3.3]). We need to make use of the following simple facts.

LEMMA 3.5. *Suppose* $\sum_n c_{2n} = \sum_n c_{2n+1} = 1$.

(i) *Let* $\mathbf{v} = [v_0, v_1, \dots, v_{N-1}]^t$ *and let* $g = \sum_{n=0}^{N-1} v_n \chi_{[n,n+1)}$. *Then for any* $k \in \mathbb{N}$ *and for almost all* $x \in [0,1)$,

$$\sum_{n=0}^{N-1} S^k g(x+n) = \sum_{n=0}^{N-1} v_n.$$

(ii) *Let* $H = \{\mathbf{u} \in \mathbb{C}^{2N-1} : \sum_{n=-(N-1)}^{N-1} u_n = 0\}$. *Then* $(\mathbf{W}_{N-1})^*$ *is invariant on* H.

Proof. The proofs of (i) and (ii) are quite similar. For (i), we make use of the fact that $[1, \dots, 1]$ is a left 1-eigenvector of $\mathbf{T}_0$ and $\mathbf{T}_1$ (see, e.g., [H]). To prove (ii), note that $[1, 1, \dots, 1]^t$ is a 2-eigenvector of $\mathbf{W}_{N-1}$ (Proposition 2.2); hence for $\mathbf{u} \in H$,

$$[1, 1, \dots, 1]\,(\mathbf{W}_{N-1})^*\,\mathbf{u} = 2\,[1, 1, \dots, 1]\,\mathbf{u} = 2 \sum_{n=-(N-1)}^{N-1} u_n = 0.$$

This implies that the sum of the coordinates of $(\mathbf{W}_{N-1})^* \mathbf{u}$ is zero so that $(\mathbf{W}_{N-1})^*$ is invariant on H.

COROLLARY 3.6. *Suppose* $\sum c_{2n} = \sum c_{2n+1} = 1$. *If the eigenvalues of* $(\mathbf{W}_{N-1})^*$ *restricted on* H *have moduli less than* 2, *then* (1.1) *has an* L^2_c*-solution.*

Proof. Let $\mathbf{v} = [v_0, v_1, \dots, v_{n-1}]^t$ be a 2-eigenvector of $\mathbf{T}_0 + \mathbf{T}_1$ as in Theorem 3.4. Then Lemma 3.5 implies that $\sum_{n=0}^{N-1} \tilde{g}(x+n) = 0$ for almost all $x \in [0,1)$. Since $\mathrm{supp} g \subseteq [0, N]$, we actually have $\sum_{n=-\infty}^{\infty} \tilde{g}(x+n) = 0$ for almost all $x \in [0,1)$ and hence for almost all $x \in \mathbb{R}$. Therefore,

$$\begin{aligned}
\sum_{|n|\le N-1} \mathbf{a}_n(\tilde{g}) &= \sum_{n=-\infty}^{\infty} \mathbf{a}_n(\tilde{g}) \\
&= \sum_{n=-\infty}^{\infty} \int_0^N \tilde{g}(t+n)\tilde{g}(t)\,dt \\
&= \int_0^N \left(\sum_{n=-\infty}^{\infty} \tilde{g}(t+n) \right) \tilde{g}(t)\,dt \\
&= 0,
\end{aligned}$$

and $\mathbf{a}(\tilde{g}) \in H$. This implies that the subspace $H_{\mathbf{v}}$ in Theorem 3.4 is contained in H. By assumption, $\frac{1}{2}(\mathbf{W}_{N-1})^*$ restricted on H has spectral radius less than 1, and Theorem 3.4 applies.

In [LW1, Prop. 4.6], it is proven that the converse of the above corollary is also true if we assume that 2 is a simple eigenvalue of $\mathbf{W}_{N-1}$ and $\{\mathbf{W}_{N-1}^l \mathbf{e}_1\}$ generates $\mathbb{C}^{2N-1}$; for the four-coefficient case (N=3), the above additional assumptions are always true except for the case $c_0 = c_3 = 1$. By using a long and rather complicated argument Cohen and Daubechies [CD, Thm. 4.3] also showed that the converse is true if f has the Riesz-basis property. In the following, we will give a short proof of their theorem.

Recall that a function $f \in L^2(\mathbb{R})$ is said to satisfy the *Riesz-basis property* if the sequence of functions $f_n = f(\cdot - n), n \in \mathbb{Z}$ forms a Riesz-basis for the closure of its linear span in $L^2(\mathbb{R})$, i.e., there exist $C_1, C_2 > 0$ such that

$$C_1 \sum |\alpha_n|^2 \le \| \sum \alpha_n f_n \|^2 \le C_2 \sum |\alpha_n|^2.$$

Cohen [C], Lawton [La], and Villemoes [V] have given different criteria for such a property in terms of the Fourier transformation. In particular, Villemoes showed that if an L^2_c-solution f has the Riesz-basis property, then $\sum c_{2n} = \sum c_{2n+1} = 1$. Also, assuming such a summing condition, f has the Riesz-basis property if and only if $\sum a_n(f) e^{in\omega}$ is strictly positive.

THEOREM 3.7. *Suppose* f *is a solution of* (1.1) *and has the Riesz-basis property. Then* $(\mathbf{W}_{N-1})^*$ *restricted on* H *has spectral radius less than* 2.

Proof. Since f has the Riesz-basis property, then $\sum c_{2n} = \sum c_{2n+1} = 1$ [V] so that $(\mathbf{W}_{N-1})^*$ is invariant on H. All eigenvalues of $(\mathbf{W}_{N-1})^*$ have moduli less than or equal to 2 (see Propositon 5.3 in §5). The proof will be complete if we show that $(\mathbf{W}_{N-1})^*$ does not have another 2-eigenvector other than $\mathbf{a}(f)$, which is not in H.

Note that $\sum a_n(f) e^{in\omega} = \sum |\hat{f}(\omega + 2\pi k)|^2 > 0$ by the Riesz-basis property. Suppose $\mathbf{u}$ is another 2-eigenvector of $(\mathbf{W}_{N-1})^*$. By letting $u_n = 0$ for all $|n| \ge N$, $\mathbf{u}$ is a 2-eigenvector of $\mathbf{W}^*$. By Wiener's theorem, there exists $\{r_n\}_{n=-\infty}^{\infty} \in \ell^1$ such that

$$\sum r_n e^{in\omega} = \frac{\sum u_n e^{in\omega}}{\sum a_n(f) e^{in\omega}}.$$

It follows that $\mathbf{u} = \mathbf{r} * (\mathbf{a}(f))$ and

$$\begin{aligned} u_n &= \lim_{l\to\infty} \frac{1}{2^l} \langle (\mathbf{W}^*)^l \mathbf{u}, \mathbf{e}_n \rangle \\ &= \lim_{l\to\infty} \frac{1}{2^l} \sum_k \langle r_k (\mathbf{W}^*)^l \mathbf{a}^{(k)}(f), \mathbf{e}_n \rangle \\ &= \lim_{l\to\infty} \sum_k \langle r_k \mathbf{a}^{(2^{-l}k)}(f), \mathbf{e}_n \rangle \qquad \text{(use (3.1)}') \\ &= \sum_k r_k \langle \lim_{l\to\infty} \mathbf{a}^{(2^{-l}k)}(f), \mathbf{e}_n \rangle \\ &= C \langle \mathbf{a}(f), \mathbf{e}_n \rangle, \end{aligned}$$

where $C = \sum_k r_k$. This implies that $\mathbf{u}$ is a scalar multiple of $\mathbf{a}(f)$ and the proof is complete.

Remark. The above discussion gives a simple criterion for computer to check for the Riesz-basis property of the solution f given $\{c_n\}_{n=0}^N$ with $\sum c_{2n} = \sum c_{2n+1} = 1$: first show that the 2-eigenvalue of $(\mathbf{W}_{N-1})^*$ is simple and all other eigenvalues are less than 2 in modulus (this implies the existence of the solution by Corollary 3.6 and $\sum a_n(f) e^{in\omega} \geq 0$ by Proposition 3.2), and then show that the polynomial $\frac{1}{2}\sum a_n(f) z^n$ has no root on the unit circle.

4. Some lemmas. In the rest of the paper, we will use the difference quotient $\frac{1}{h^{2\beta}} \int_{-\infty}^{\infty} |\Delta_h f(t)|^2 dt$ to study the regularity properties of the scaling function f. We prefer to use the matrix $\mathbf{W}_N^+$ rather than $\mathbf{W}_N$ because using the latter, we have to discard those eigenvalues which only give antisymmetric eigenvectors (see the remark for Proposition 2.1 and Lemma 4.3). As before, we assume that $\sum c_n = 2$. For $h \in \mathbb{R}$ and $n \in \mathbb{Z}$, we also define

$$\Phi_n(h) = \int_{-\infty}^{\infty} \Delta_h f(t+n) \Delta_h f(t) dt.$$

Since f is supported by $[0, N]$,

$$\Phi_n(h) = 0 \quad \forall\, 0 < h < 1,\ |n| \geq N+1. \tag{4.1}$$

We use $\mathbf{\Phi}(h)$ to denote

$$[\Phi_0(h), \Phi_1(h), \ldots, \Phi_N(h)]^t \quad \text{and} \quad [\Phi_0(h), \Phi_1(h), \ldots]^t.$$

If necessary, we will add the superscript N or ∞ to $\mathbf{\Phi}(h)$ to make the distinction. It is clear that $\Delta_h f$ satisfies

$$\Delta_h f(x) = \sum_{n=0}^{N} c_n \Delta_{2h} f(2x - n). \tag{4.2}$$

PROPOSITION 4.1. *Let* $\mathbf{W}$ *be the transition matrix corresponding to the scaling function* f *satisfying* (1.1). *Then for* $\mathbf{u} = [u_0, u_1, \ldots, u_N]^t \in \mathbb{C}^{N+1}$,

$$\langle \mathbf{\Phi}(h), \mathbf{u} \rangle = \frac{1}{2} \langle \mathbf{\Phi}(2h), \mathbf{W}_N^+ \mathbf{u} \rangle, \quad 0 < h < \frac{1}{2}. \tag{4.3}$$

Proof. The proof is basically the same as that of Proposition 3.1, using (4.2) instead of (1.1). For $0 \le n \le N$,

$$\begin{aligned}
\Phi_n(h) &= \int_{-\infty}^{\infty} \Delta_h f(t+n) \Delta_h f(t)\, dt \\
&= \sum_{i,j \in \mathbb{Z}} c_j c_i \int_{-\infty}^{\infty} \Delta_{2h} f(2t + 2n - j) \Delta_{2h} f(2t - i)\, dt \\
&\ \ \vdots \\
&= \frac{1}{2} [\mathbf{W}^* \mathbf{G}^* (\mathbf{\Phi}(2h))]_n \\
&= \frac{1}{2} [(\mathbf{W}_N^+)^* (\mathbf{\Phi}(2h))]_n.
\end{aligned}$$

LEMMA 4.2. *If* $\mathbf{u}$ *is a 2-eigenvector or a 0-eigenvector of* $\mathbf{W}_N^+$, *then*

$$\langle \mathbf{\Phi}(h), \mathbf{u} \rangle = 0 \qquad \forall\, 0 < h < 1.$$

Proof. Let $\mathbf{u}$ be a 2-eigenvector of $\mathbf{W}_N^+$. We have, by (4.3), $\langle \mathbf{\Phi}(\frac{h}{2}), \mathbf{u} \rangle = \langle \mathbf{\Phi}(h), \mathbf{u} \rangle$. for all $0 < h < 1$. Hence, inductively,

$$\langle \mathbf{\Phi}(h), \mathbf{u} \rangle = \left\langle \mathbf{\Phi}\left(\frac{h}{2^2}\right), \mathbf{u} \right\rangle = \cdots = \left\langle \mathbf{\Phi}\left(\frac{h}{2^m}\right), \mathbf{u} \right\rangle$$

for all $0 < h < 1$, $m \ge 0$. But $\langle \mathbf{\Phi}(\frac{h}{2^m}), \mathbf{u} \rangle \to 0$ as $m \to \infty$, so it follows that $\langle \mathbf{\Phi}(h), \mathbf{u} \rangle = 0$ for all $0 < h < 1$. The same conclusion also holds if $\mathbf{u}$ is a 0-eigenvector since in such a case $\mathbf{W}_N^+ \mathbf{u} = 0$.

Our main lemma is the following.

LEMMA 4.3. *If* $\mathbf{u}$ *is a* λ*-eigenvector of* $\mathbf{W}_N^+$ *with* $\lambda \ne 0$ *or* 2, *then*

$$\langle \mathbf{\Phi}(h), \mathbf{u} \rangle \ne 0 \qquad \text{for some } 0 < h < \frac{1}{2}.$$

The proof of this lemma is rather long. The basic idea is to prove by contradiction. Suppose otherwise, i.e., $\psi(h) = \langle \mathbf{\Phi}(h), \mathbf{u} \rangle = 0$ for all $0 < h < \frac{1}{2}$. We show that when $\mathbf{u}$ is replaced by the corresponding λ-eigenvector $\tilde{\mathbf{u}}$ for $\mathbf{W}^+$ (Lemma 4.4) and the inner product in $\psi(h)$ is acting on all positive coordinates, then the identity holds for all $h \in \mathbb{R}$. From this we deduce that the corresponding sequence $\{u_n\}$ is a linear combination of certain exponential sequences of the form $\{e^{ian}\}$. However, the eigenproperty of $\{u_n\}$ implies that this is impossible.

For this purpose, we first observe that for $h \in \mathbb{R}$,

$$\begin{aligned}
\Phi_n(h) &= \int_{-\infty}^{\infty} \Delta_h f(t+n) \Delta_h f(t)\, dt \\
&= \int_{-\infty}^{\infty} (f(t+n+h) - f(t+n))(f(t+h) - f(t))\, dt.
\end{aligned}$$

Multiplying out the integrand and changing the variables, we obtain

$$\Phi_n(h) = 2a_n(f) - \Psi_n(h), \tag{4.4}$$

where

$$\Psi_n(h) = \int_{-\infty}^{\infty} [f(t+h-n) + f(t+h+n)]f(t)\,dt$$

and $a_n(f)$ is the nth autocorrelation. We let

$$\Psi^N(h) = [\Psi_0(h), \Psi_1(h), \dots, \Psi_N(h)]^t \quad \text{and} \quad \Psi^\infty(h) = [\Psi_0(h), \Psi_1(h), \dots]^t.$$

Note that $[a_0(f), a_1(f), \dots, a_N(f)]^t$ is a 2-eigenvector of $(\mathbf{W}_N^+)^*$ (use Proposition 3.2 and (2.2)). It is orthogonal to any λ-eigenvector $\mathbf{u}$ of $\mathbf{W}_N^+$ with $\lambda \neq 2$. For such $\mathbf{u}$, (4.4) implies that

$$\langle \mathbf{\Phi}^N(h), \mathbf{u} \rangle = -\langle \mathbf{\Psi}^N(h), \mathbf{u} \rangle, \quad h \in \mathbb{R}. \tag{4.5}$$

Let $\mathcal{S}$ be the class of all one-sided infinite sequences.

LEMMA 4.4. *If* $\mathbf{u}$ *is a* λ*-eigenvector of* $\mathbf{W}_N^+$ *with* $\lambda \neq 0$, *then there exists* $\tilde{\mathbf{u}} \in \mathcal{S}$, *a* λ*-eigenvector of* $\mathbf{W}^+$ *such that* $\tilde{u}_n = u_n$ *for all* $0 \le n \le N$. *Furthermore, for such a* $\tilde{\mathbf{u}}$,

$$\langle \mathbf{\Psi}^\infty(h), \tilde{\mathbf{u}} \rangle = \langle \mathbf{\Psi}^N(h), \mathbf{u} \rangle, \quad 0 < h < \frac{1}{2}.$$

Proof. Let $\mathbf{u} = [u_0, u_1, \dots, u_N]^t$ be a λ-eigenvector of $\mathbf{W}_N^+$ with $\lambda \neq 0$. Note that for $i > 0$,

$$w_{ij}^+ = \omega_{i-2j} + \omega_{-i-2j}.$$

Since $\omega_n = 0$ for all $|n| > N$ and $N < i \le j$, $-i-2j < i-2j < -N$, we hence have $w_{ij}^+ = 0$ for all $N < i \le j$. We now construct $\tilde{\mathbf{u}}$ as follows: let $\tilde{u}_n = u_n$ for all $0 \le n \le N$ and define $\tilde{u}_{n+1}$ inductively as

$$\tilde{u}_{n+1} = \frac{1}{\lambda} \sum_{k=0}^{n} w_{n+1,k}^+ \tilde{u}_k, \quad n \ge N. \tag{4.6}$$

Then $\tilde{\mathbf{u}}$ is the required vector. The last assertion follows from the fact that for $0 < h < \frac{1}{2}$, $\Psi_n(h) = 0$ for all $n > N$. ∎

In [Sch] (see also [K], [RL]), L. Schwartz proved the following classical result on *mean periodic functions*: Let μ be a bounded regular Borel measure on $\mathbb{R}$ with compact support. Let $\mathcal{C}$ be the class of continuous functions on $\mathbb{R}$ equipped with the compact open topology. Suppose there exists a nonzero $g \in \mathcal{C}$ that satisfies the convolution equation

$$\int_{-\infty}^{\infty} g(x-y)\,d\mu(y) = 0 \qquad \forall\, x \in \mathbb{R}.$$

Then g belongs to the closed linear subspace spanned by

$$\{e^{ia(\cdot)} : a \in \mathbb{C}, \int_{-\infty}^{\infty} e^{-iay}\,d\mu(y) = 0\}.$$

Heuristically, the convolution equation implies that $\hat{g}(z)\hat{\mu}(z) = 0$ (in the distribution sense). Since μ has compact support, $\hat{\mu}$ is an entire function and has only countably many discrete zeros. It follows that the support of $\hat{g}$ must be contained in the zeros of $\hat{\mu}$, and g is of the form asserted. We need the discrete version, which is an easy corollary of the above theorem: if $\{w_n\}_{n=-\infty}^{\infty}$ is a given sequence with only finitely many nonzero terms and if $\{x_n\}_{n=-\infty}^{\infty}$ is any sequence satisfying

$$\sum_{k=-\infty}^{\infty} x_{n-k}w_k = 0 \quad \forall\, n \in \mathbb{Z},$$

then $\{x_n\}_{n=-\infty}^{\infty}$ belongs to the closed (with respect to the product topology) linear subspace spanned by

$$\{\{e^{ian}\} : a \in \mathbb{C},\ \sum_{n=-\infty}^{\infty} e^{-ian}w_n = 0\}.$$

LEMMA 4.5. *Let* $\tilde{\mathbf{u}}$ *be a* λ*-eigenvector of* $\mathbf{W}^+$ *with* $\lambda \neq 0$ *or* 2. *Then*

$$\langle \mathbf{\Psi}^\infty(h), \tilde{\mathbf{u}}\rangle \neq 0 \qquad \text{for some } 0 < h < \frac{1}{2}.$$

Proof. By Proposition 4.1, Lemma 4.4, and (4.5), we have for any $\mathbf{v} \in \mathcal{S}$,

$$\langle \mathbf{\Psi}^\infty(h), \mathbf{v}\rangle = \frac{1}{2}\langle \mathbf{\Psi}^\infty(2h), \mathbf{W}^+\mathbf{v}\rangle \quad \forall\, h \in \mathbb{R}. \tag{4.7}$$

For any fixed h, $\Psi_n(h) = 0$ for all large n; hence $\langle \mathbf{\Psi}^\infty(h), \tilde{\mathbf{u}}\rangle$ is well defined and is continuous on h. Suppose the lemma is false, i.e.,

$$\langle \mathbf{\Psi}^\infty(h), \tilde{\mathbf{u}}\rangle = 0 \qquad \forall\, 0 < h < \frac{1}{2}. \tag{4.8}$$

By (4.7), we have

$$0 = \langle \mathbf{\Psi}^\infty(h), \tilde{\mathbf{u}}\rangle = \frac{1}{2}\langle \mathbf{\Psi}^\infty(2h), \mathbf{W}^+\tilde{\mathbf{u}}\rangle = \frac{\lambda}{2}\langle \mathbf{\Psi}^\infty(2h), \tilde{\mathbf{u}}\rangle.$$

The assumption that $\lambda \neq 0$ implies that $\langle \mathbf{\Psi}^\infty(2h), \tilde{\mathbf{u}}\rangle = 0$ for all $0 < h < \frac{1}{2}$, i.e., (4.8) holds for all $0 < h < 1$. Repeating the same argument, we have that (4.8) holds for all $h > 0$ and hence for all $h \in \mathbb{R} \setminus \{0\}$ since $\mathbf{\Psi}^\infty(-h) = \mathbf{\Psi}^\infty(h)$. By continuity, we also have $\langle \mathbf{\Psi}^\infty(0), \tilde{\mathbf{u}}\rangle = 0$. We hence conclude that

$$\sum_{k=0}^{\infty} \tilde{u}_k \left(f * \tilde{f}(h-k) + f * \tilde{f}(h+k)\right) = 0 \qquad \forall\, h \in \mathbb{R},$$

where $\tilde{f}(x) = f(-x)$. By letting $x_0 = 2u_0$, $x_n = x_{-n} = \tilde{u}_n$ for $n > 0$, and by replacing h with $h + n$, $0 \leq h < 1$, we can rewrite the above as

$$\sum_{k=-\infty}^{\infty} x_{n-k} f * \tilde{f}(h+k) = 0 \qquad \forall\, n \in \mathbb{Z},\ h \in [0,1).$$

Note that the autocorrelation function $f * \tilde{f}$ is continuous and has compact support. For each fixed $h \in [0,1)$, if we regard the sequence $\{f * \tilde{f}(h+n)\}$ as the $\{w_n\}$ in the above digression, then $\{x_n\}_{n=-\infty}^{\infty}$ must be in the closed linear subspace spanned by

$$A_h = \{\{e^{ian}\} : a \in \mathbb{C}, \sum_{n=-\infty}^{\infty} f * \tilde{f}(h+n)e^{-ian} = 0\}.$$

Since this is true for all $h \in [0,1)$, $\{x_n\}_{n=-\infty}^{\infty}$ must be in the closed linear subspace spanned by $\bigcap_{h\in[0,1)} A_h$. By using the Poisson summation formula ([Ch, p. 47]), we have

$$\begin{aligned} 0 &= \sum_{n=-\infty}^{\infty} f * \tilde{f}(h+n)e^{-ina} \\ &= \sum_{n=-\infty}^{\infty} \hat{f}(a+2\pi n)\hat{f}(-(a+2\pi n))e^{ih(a+2\pi n)} \\ &= e^{iha} \sum_{n=-\infty}^{\infty} \hat{f}(a+2\pi n)\hat{f}(-(a+2\pi n))e^{i2\pi nh} \end{aligned}$$

for all $h \in [0,1)$. This implies that

$$\hat{f}(a+2\pi n)\hat{f}(-(a+2\pi n)) = 0 \quad \forall n \in \mathbb{Z}.$$

Observe that the Fourier transformation of (1.1) is $\hat{f}(z) = \hat{f}(\frac{z}{2})m_0(\frac{z}{2})$, where $m_0(z) = \frac{1}{2}\sum c_n e^{inz}$ is a trigonometric polynomial of degree N. Let $F(z) = \hat{f}(z)\hat{f}(-z)$, $Q(e^{iz}) = m_0(z)m_0(-z)$. Since $F \neq 0$ in a neighborhood of 0, we conclude from $0 = F(a) = F(\frac{a}{2})Q(e^{ia/2})$ that for some l, $e^{ia/2^l}$ must be a root of Q. Hence the sequence $\{x_n\}_{n=-\infty}^{\infty}$ is in the close linear span of all the sequences of the form

$$\{\{e^{ian}\} : e^{ia/2^l} \text{ is a root of } Q(z) \text{ for some } l\}. \tag{4.9}$$

Now, by a direct calculation,

$$[\mathbf{W}(e^{ia(\cdot)})]_n = \sum_{k=-\infty}^{\infty} w_{n-2k}e^{iak} = 2^2 e^{ian/2}Q(e^{ia/2}).$$

This implies that for some l,

$$\mathbf{W}^l(e^{ia(\cdot)}) = 2^{2l}e^{ia(\cdot)/2^l}Q(e^{ia/2^l}) = 0. \tag{4.10}$$

On the other hand, in view of Lemma 4.4, the vector $\mathbf{x} = [\ldots, x_{-1}, x_0, x_1, \ldots]$ satisfies $\mathbf{W}\mathbf{x} = \lambda\mathbf{x}$. This is a contradiction since $\{x_n\}_{n=-\infty}^{\infty}$ is a combination of the sequences $\{e^{ian}\}_{n=-\infty}^{\infty}$ in (4.9), and (4.10) implies that $\mathbf{x}$ can not be an eigenvector.

Proof of Lemma 4.3. Suppose that $\mathbf{u}$ is an λ-eigenvector of $\mathbf{W}_N^+$ with $\lambda \neq 0$ or 2. By Lemma 4.4, there exists $\tilde{\mathbf{u}} \in \mathcal{S}$ such that $\mathbf{W}^+\tilde{\mathbf{u}} = \lambda\tilde{\mathbf{u}}$ and $\tilde{u}_n = u_n$ for all $0 \leq n \leq N$. By Lemma 4.5, we have $\langle \mathbf{\Psi}^\infty(h), \tilde{\mathbf{u}}\rangle \neq 0$ for some $0 < h < \frac{1}{2}$. For such h,

$$\begin{aligned} \langle \mathbf{\Phi}^N(h), \mathbf{u}\rangle &= -\langle \mathbf{\Psi}^N(h), \mathbf{u}\rangle && \text{(by (4.5))} \\ &= -\langle \mathbf{\Psi}^\infty(h), \tilde{\mathbf{u}}\rangle \\ &\neq 0. \end{aligned}$$

5. The L^2-Lipschitz exponent and asymptotics. For any $\beta \in \mathbb{C}$, we let

$$\mathbf{\Phi}^{(\beta)}(h) = \frac{1}{h^{2\beta}}\mathbf{\Phi}(h).$$

LEMMA 5.1. *Let $\lambda \neq 0, 2$ be an eigenvalue of $\mathbf{W}_N^+$, let $\beta = -\ln(\lambda/2)/(2\ln 2)$ (i.e., $\lambda/2^{1-2\beta} = 1$ and β takes the principal branch when λ is complex), and let $\hat{\mathbf{u}}$ be the corresponding eigenvector. Then*

$$\langle \mathbf{\Phi}^{(\beta)}(2h), \hat{\mathbf{u}}\rangle = \langle \mathbf{\Phi}^{(\beta)}(h), \hat{\mathbf{u}}\rangle \quad \forall\, 0 < h < \frac{1}{2}.$$

Proof. Let $\phi(h) = \langle \mathbf{\Phi}^{(\beta)}(h), \hat{\mathbf{u}}\rangle$. By Proposition 4.1, we have for $0 < h < \frac{1}{2}$,

$$\begin{aligned}\langle \mathbf{\Phi}^{(\beta)}(h), \hat{\mathbf{u}}\rangle &= \frac{1}{h^{2\beta}}\langle \mathbf{\Phi}(h), \hat{\mathbf{u}}\rangle \\ &= \frac{1}{2^{1-2\beta}} \cdot \frac{1}{(2h)^{2\beta}}\langle \mathbf{\Phi}(2h), \mathbf{W}_N^+\hat{\mathbf{u}}\rangle \\ &= \frac{\lambda}{2^{1-2\beta}}\langle \mathbf{\Phi}^{(\beta)}(2h), \hat{\mathbf{u}}\rangle.\end{aligned}$$

By the choice of β, we have $\phi(h) = \phi(2h)$ for all $0 < h < \frac{1}{2}$.

Recall that if $\mathbf{M}$ is a matrix on a vector space V with characteristic polynomial $p(x) = (x - \lambda_1)^{\ell_1} \cdots (x - \lambda_k)^{\ell_k}$ and minimal polynomial $q(x) = (x - \lambda_1)^{m_1} \cdots (x - \lambda_k)^{m_k}$, then $V = V_1 \oplus \cdots \oplus V_k$, each V_i has dimension ℓ_i, $\mathbf{M}$ is invariant on V_i, and $(\mathbf{M} - \lambda_i\mathbf{I})^{m_i}V_i = 0$ (m_i is called the *order* of λ_i). Moreover, according to the Jordan decomposition theorem,

$$V_i = U_{i1} \oplus \cdots \oplus U_{ir_i},$$

where each $s_{ij} := \dim U_{ij} \leq m_i$, with at least one of the $s_{ij} = m_i$; each U_{ij} is generated by

$$\mathbf{u}_1 = \mathbf{u},\ \mathbf{u}_2 = (\mathbf{M} - \lambda_i\mathbf{I})\mathbf{u}, \ldots,\ \mathbf{u}_{s_{ij}} = (\mathbf{M} - \lambda_i\mathbf{I})^{s_{ij}-1}\mathbf{u} \tag{5.1}$$

and $(\mathbf{M}-\lambda_i\mathbf{I})^{s_{ij}}\mathbf{u} = 0$ for some $\mathbf{u}$. Note that the last vector in (5.1) is a λ_i-eigenvector of $\mathbf{M}$.

Lemma 5.1 can be strengthened as follows.

THEOREM 5.2. *Let $\lambda \neq 0, 2$ be an eigenvalue of $\mathbf{W}_N^+$ and let $\beta = -\ln(\lambda/2)/(2\ln 2)$. Suppose there exists an m such that $(\mathbf{W}_N^+ - \lambda\mathbf{I})^{m-1}\mathbf{u} \neq \mathbf{0}$, $(\mathbf{W}_N^+ - \lambda\mathbf{I})^{m}\mathbf{u} = \mathbf{0}$. Then*

$$\langle \mathbf{\Phi}^{(\beta)}(h), \mathbf{u}\rangle = \sum_{k=1}^{m}(\ln h)^{k-1}p_k(h), \quad 0 < h < \frac{1}{2}, \tag{5.2}$$

where $p_k(h) = p_k(2h)$ for all $h > 0$ and $p_m \neq 0$. In particular, if $m = 1$, then $\langle \mathbf{\Phi}^{(\beta)}(h), \mathbf{u}\rangle = p_1(h)$.

Proof. Let

$$\mathbf{u}_m = \mathbf{u}, \ldots,\ \mathbf{u}_1 = (\mathbf{W}_N^+ - \lambda\mathbf{I})^{m-1}\mathbf{u}$$

and let $\phi_k(h) = \langle \mathbf{\Phi}^{(\beta)}(h), \mathbf{u}_k\rangle$. Note that $\mathbf{u}_1$ is a λ-eigenvector of $\mathbf{W}_N^+$. Hence by Lemma 4.3, $\phi_1 \neq 0$, and Lemma 5.1,

$$\phi_1(h) = \phi_1(2h) \qquad \forall\, 0 < h < \frac{1}{2}.$$

Let $g_1(h) = \phi_1(h)$. For $\mathbf{W}_N^+\mathbf{u}_2 = \lambda\mathbf{u}_2 + \mathbf{u}_1$, by the same argument as in Lemma 5.1, we have

$$\phi_2(h) = \phi_2(2h) + \frac{1}{\lambda}\phi_1(2h).$$

Let $g_2(h) = \phi_2(h) + \frac{\ln h}{\lambda \ln 2} g_1(h)$. Then

$$\phi_2(h) = g_2(h) - \frac{\ln h}{\lambda \ln 2} g_1(h),$$

and for $0 < h < \frac{1}{2}$,

$$\begin{aligned} g_2(h) &= \phi_2(2h) + \frac{1}{\lambda}\phi_1(h) + \frac{\ln h}{\lambda \ln 2}\, g_1(h) \\ &= \phi_2(2h) + \frac{\ln(2h)}{\lambda \ln 2} g_1(2h) \\ &= g_2(2h). \end{aligned}$$

Let $g_3(h) = \phi_3(h) + \frac{\ln h}{\lambda \ln 2} g_2(h) - \frac{(\ln h)(\ln(2h))}{2(\lambda \ln 2)^2} g_1(h)$. Then by a similar argument as above, we have

$$g_3(h) = g_3(2h) \qquad \forall\, 0 < h < \frac{1}{2}.$$

Inductively, we can find g_j, $1 \le j \le m$, such that $g_j(h) = g_j(2h)$ for $0 < h < \frac{1}{2}$ and

$$\phi_j(h) = g_j(h) + \sum_{k=1}^{j-1} \frac{(-1)^{j-k}}{(j-k)!\,(\lambda \ln 2)^{j-k}} \left(\prod_{l=1}^{j-k} \ln(2^{l-1}h) \right) g_k(h).$$

For $j = m$, we group those terms with $(\ln h)^k$ together and denote the corresponding coefficient by $p_k(h)$. Then p_k satisfies the periodic condition, and $p_m(h) = c\phi_1(h) \neq 0$. If we extend p_k by $p_k(h) = p_k(2h)$ to all h, the theorem follows.

Now, we define

$$\Lambda_{\max} = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathbf{W}_N^+ \text{ and } |\lambda| \neq 2\}.$$

PROPOSITION 5.3. *Suppose f is an L_c^2-solution of* (1.1). *Then* $\frac{1}{2} \le \Lambda_{\max} < 2$.

Proof. We first claim that $\Lambda_{\max} < 2$. Otherwise, let λ be an eigenvalue with $|\lambda| > 2$ and let $\mathbf{u}$ be a corresponding eigenvector. By using Proposition 4.1, we have

$$\left\langle \Phi\left(\frac{h}{2^m}\right), \mathbf{u} \right\rangle = \left(\frac{\lambda}{2}\right)^m \langle \Phi(h), \mathbf{u} \rangle \quad \forall\, 0 < h < \frac{1}{2}.$$

By Lemma 4.3, there exists $0 < h < \frac{1}{2}$ such that $|\langle \Phi(h), \mathbf{u}\rangle| \neq 0$. Hence $|\langle \Phi(\frac{h}{2^m}), \mathbf{u}\rangle|$ does not tend to zero as $m \to \infty$. This is a contradiction, and the claim follows.

If $\Lambda_{\max} < \frac{1}{2}$, then for any $\mathbf{u}$ in Theorem 5.2, the corresponding β satisfies $\operatorname{Re}\beta = -\ln(|\lambda|/2)/(2\ln 2) > 1$ and hence $\limsup_{h\to 0^+} \frac{1}{h^2} |\langle \Phi(h), \mathbf{u}\rangle| = 0$. Since all such $\mathbf{u}$ form a Jordan basis, it follows that

$$\limsup_{h\to 0^+} \frac{1}{h^2} \int_{-\infty}^{\infty} |\Delta_h f(t)|^2\, dt = \limsup_{h\to 0^+} \frac{1}{h^2} \langle \Phi(h), \mathbf{e}_0 \rangle = 0.$$

This implies that $\sup_{h>0} \frac{1}{h^2} \int_{-\infty}^{\infty} |\Delta_h f|^2 < \infty$ so that $f' \in L^2(\mathbb{R})$ and $\int |f'|^2 = \lim_{h\to 0} \frac{1}{h^2} \int_{-\infty}^{\infty} |\Delta_h f|^2 = 0$. This implies that $f = 0$ a.e., a contradiction.

THEOREM 5.4. *Suppose f is an L^2_c-solution of the dilation equation* (1.1). *Let $\alpha = -\ln(\Lambda_{\max}/2)/(2\ln 2)$ and let m be the highest order of the eigenvalues λ of $\mathbf{W}_N^+$ such that $|\lambda| = \Lambda_{\max}$. Then*

$$\lim_{h\to 0^+}\left(\frac{1}{h^{2\alpha}|\ln h|^{m-1}}\int_{-\infty}^{\infty}|\Delta_h f(x)|^2dx \;-\; p(h)\right) = 0, \tag{5.3}$$

where $p(h)$ is a nonzero bounded continuous multiplicative periodic function of period 2, *i.e., $p(h) = p(2h), h > 0$.*

Proof. Write $\mathbf{e}_0 = \sum_i b_i\mathbf{u}_i$, where $\{\mathbf{u}_i\}$ is a Jordan basis corresponding to the matrix $\mathbf{W}_N^+$. Let λ be the eigenvalues (there may be more than one) such that $|\lambda| = \Lambda_{\max}$ and has highest order m. By Theorem 5.2 and the choice of α, the terms $|\langle\mathbf{\Phi}(h), \mathbf{u}_i\rangle|$ of the form $h^{2\alpha}|\ln h|^{m-1}p(h)$ dominate $\langle\mathbf{\Phi}(h), \mathbf{e}_0\rangle$ as $h \to 0^+$; the corresponding coefficient b_i's are not all zero since $|\langle\mathbf{\Phi}(h), \mathbf{e}_0\rangle| \geq c|\langle\mathbf{\Phi}(h), \mathbf{u}_i\rangle|$ for some $c > 0$. We hence have

$$\int_{-\infty}^{\infty}|\Delta_h f(x)|^2dx = \langle\mathbf{\Phi}(h), \mathbf{e}_0\rangle = h^{2\alpha}|\ln h|^{m-1}p(h) + \delta(h),$$

where $p(h) = p(2h)$ and $\lim_{h\to 0^+}\delta(h)/(h^{2\alpha}|\ln h|^{m-1}) = 0$, and the theorem follows.

COROLLARY 5.5. *Suppose λ is an eigenvalue of $\mathbf{W}_N^+$ such that $|\lambda| = \Lambda_{\max} = \frac{1}{2}$ and λ has order* 1. *Then $\lambda = \frac{1}{2}$, f is differentiable a.e., and $f \in L^2(\mathbb{R})$.*

Proof. Let $\beta = -\ln(\lambda/2)/(2\ln 2) = 1 + i\theta$ and let $\mathbf{u}$ be the λ-eigenvector. Then

$$\begin{aligned}\frac{1}{h^2}\int_{-\infty}^{\infty}|\Delta_h f(x)|^2dx &= h^{i2\theta}\langle\mathbf{\Phi}^{\beta}(h), \mathbf{u}\rangle + o(h)\\ &= h^{i2\theta}p(h) + o(h),\end{aligned}$$

where $o(h) \to 0$ as $h \to 0$ and p is bounded and $p(h) = p(2h)$. It is well known that if $\sup_{h>0}\frac{1}{h^2}\int_{-\infty}^{\infty}|\Delta_h f(x)|^2dx$ is bounded, then f' exists a.e., $f' \in L^2(\mathbb{R})$, and

$$\lim_{h\to 0^+}\frac{1}{h^2}\int_{-\infty}^{\infty}|\Delta_h f(x)|^2dx = \int_{-\infty}^{\infty}|f'(x)|^2dx.$$

This implies that $\lim_{h\to 0^+}h^{i2\theta}p(h)$ exists. In view of periodicity, we have $\theta = 0$ and $p(h) = C$, and the corollary follows.

COROLLARY 5.6. *Let f, α, and m be as in Theorem* 5.4. *Then*

$$\sup_{n>0}\frac{1}{n^{m-1}2^{-2n\alpha}}\int_{2^{n-1}\pi\leq|\omega|<2^n\pi}|\hat{f}(\omega)|^2d\omega < \infty.$$

Proof. By using the Plancherel Theorem, we have

$$\begin{aligned}\varphi(h) &:= \frac{1}{h^{2\alpha}|\ln h|^{m-1}}\int_{-\infty}^{\infty}|\Delta_h f(x)|^2dx\\ &= \frac{C_1}{h^{2\alpha}|\ln h|^{m-1}}\int_{-\infty}^{\infty}|\hat{f}(\omega)|^2\sin^2(h\omega/2)d\omega\\ &\geq \frac{C_2}{h^{2\alpha}|\ln h|^{m-1}}\int_{\frac{\pi}{2h}\leq|\omega|<\frac{\pi}{h}}|\hat{f}(\omega)|^2d\omega.\end{aligned}$$

Since $\varphi(h)$ is bounded by Theorem 5.4, the result follows by taking $h = 2^{-n}$.

We remark that the above corollary implies that for any $r < \alpha$, f is in the Bosov space $B_2^{r,\infty}$ in the sense used in [CD, Thm. 3.3]. It also implies that $\int_{-\infty}^{\infty} |\omega^r \hat{f}(\omega)|^2 d\omega < \infty$ for $r < \alpha$, which is proven in [V]. By using the same technique as in [CD, Thm. 5.1], we can improve a pointwise estimate of $\hat{f}(\omega)$ presented there.

COROLLARY 5.7. *Let f, α, and m be as in Theorem* 5.4 *and assume that* $\sum c_{2n} = \sum c_{2n+1} = 1$. *Then*

$$|\hat{f}(\omega)| \leq \frac{C(\ln(1+|\omega|))^{(m-1)/2}}{(1+|\omega|)^{\alpha}}.$$

Proof. For $\omega \in [2^{n-1}\pi, 2^n\pi], n \geq 1$, the assumption on the coefficients implies that $\hat{f}(2k\pi) = 0$ [Ch]. Hence

$$\begin{aligned}
|\hat{f}(\omega)|^2 &\leq \int_{2^{n-1}\pi \leq |\xi| \leq 2^n\pi} \frac{d|\hat{f}(\xi)|^2}{d\xi} d\xi \\
&\leq C_1 \left(\int |\frac{d\hat{f}(\xi)}{d\xi}|^2 d\xi \right)^{1/2} \left(\int_{2^{n-1}\pi \leq |\xi| \leq 2^n\pi} |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \\
&\leq C_2 n^{(m-1)/2} 2^{-n\alpha} \\
&\leq \frac{C_3(\ln(1+|\omega|))^{(m-1)/2}}{(1+|\omega|)^{\alpha}}.
\end{aligned}$$

We define the L^2*-Lipschitz exponent* of a function $g \in L^2(\mathbb{R})$ as

$$\alpha := L^2\text{-Lip}(g) = \inf\{\beta > 0 : 0 < \limsup_{h \to 0^+} \frac{1}{h^{2\beta}} \int_{-\infty}^{\infty} |\Delta_h g(t)|^2 \, dt\}. \tag{5.4}$$

Note that $0 < \limsup_{h\to 0^+} \frac{1}{h^2} \int_{-\infty}^{\infty} |\Delta_h g(t)|^2 \, dt$ (otherwise, we can derive a contradiction by using the argument in the last paragraph of Proposition 5.3 to show that $g = 0$ a.e.). Hence $0 \leq \alpha \leq 1$. Also,

$$\limsup_{h \to 0^+} \frac{1}{h^{2\beta}} \int_{-\infty}^{\infty} |\Delta_h g(t)|^2 \, dt = \begin{cases} 0 & \text{if } \beta < \alpha, \\ \infty & \text{if } \beta > \alpha. \end{cases}$$

The next corollary follows directly from Theorem 5.4.

COROLLARY 5.8. *Suppose f is an L_c^2-solution of the dilation equation* (1.1). *Let* $\alpha = -\ln(\Lambda_{\max}/2)/(2\ln 2)$. *Then $0 < \alpha \leq 1$ is the $L^2(\mathbb{R})$-Lipschitz exponent of f.*

Corollaries 5.6 and 5.7 give certain estimates of the Fourier transform of f. In the following we consider yet another sharper estimate on the average of the Fourier transformation of the L^2-scaling function. We make use of a special form of Tauberian theorem to convert the asymptotic result in Theorem 5.4 into the frequency domain. For $\beta, \gamma \in \mathbb{R}$, let

$$\mathcal{W}_{\beta,\gamma} = \left\{ g : g \text{ loc. Riem. integ. on } \mathbb{R}^+, \sum_{k=-\infty}^{\infty} \sup_{2^k \leq t < 2^{k+1}} t^{\beta} |\ln t|^{\gamma} |g(t)| < \infty \right\}.$$

The following theorem is proven in [L3, Cor. 4.5].

THEOREM 5.9. *Suppose $F \geq 0$ is measurable on $\mathbb{R}^+$ and is bounded on $[0, a)$ for some $a > 0$. Let $g \in \mathcal{W}_{\beta,\gamma}(\mathbb{R}^+), \beta > 0, \gamma \geq 0$ be such that $G(\xi) = \int_0^\infty g(t)t^{\beta-1+i\xi}dt \neq 0$ for all ξ. Then*

$$\lim_{T\to\infty} \left(\frac{1}{T^\beta (\ln T)^\gamma} \int_0^\infty F(t) g\left(\frac{t}{T}\right) dt - P(T) \right) = 0$$

if and only if

$$\lim_{T\to\infty} \left(\frac{1}{T^\beta (\ln T)^\gamma} \int_0^T F(t)dt - Q(T) \right) = 0,$$

where P and Q are bounded multiplicative periodic functions of the same period and $P \not\equiv 0$ if and only if $Q \not\equiv 0$.

THEOREM 5.10. *Suppose f is the L_c^2-solution of (1.1) with L^2-Lipschitz exponent $\alpha \neq 1$. Let m be the highest order of the eigenvalues λ such that $|\lambda| = \Lambda_{\max}$. Then for any s such that $\alpha < s$, there exists a bounded continuous multiplicative periodic function q such that $q(T) = q(2T)$ and*

$$\lim_{T\to\infty} \left(\frac{1}{T^{2(s-\alpha)} (\ln T)^{m-1}} \int_{-T}^T |\xi^s \hat{f}(\xi)|^2 d\xi - q(T) \right) = 0.$$

Proof. By using the Phancherel Theorem as in Corollary 5.6,

$$\begin{aligned} \varphi(h) &= \frac{1}{h^{2\alpha} |\ln h|^{m-1}} \int_{-\infty}^\infty |\hat{f}(\omega)|^2 \sin^2(h\omega/2) d\omega \\ &= \frac{1}{h^{2(\alpha-s)} |\ln h|^{m-1}} \int_{-\infty}^\infty |\omega^s \hat{f}(\omega)|^2 \frac{\sin^2(h\omega/2)}{|h\omega|^{2s}} d\omega. \end{aligned}$$

By letting

$$F(\omega) = |\omega^s \hat{f}(\omega)|^2 + |\omega^s \hat{f}(-\omega)|^2, \quad g(\omega) = \frac{\sin^2(\omega/2)}{|\omega|^{2s}}, \quad h = \frac{1}{T},$$

the above reduces to

$$\varphi\left(\frac{1}{T}\right) = \frac{1}{T^{2(s-\alpha)} |\ln T|^{m-1}} \int_0^\infty F(\omega) g(\frac{\omega}{T}) d\omega.$$

Let $\beta = 2(s - \alpha)$. Then $g \in \mathcal{W}_{\beta,m-1}$. Indeed, for $0 < \alpha < 1$,

$$\begin{aligned} &\sum_{k=-\infty}^{\infty} \sup_{2^k \leq \omega < 2^{k+1}} \omega^\beta |\ln \omega|^{m-1} g(\omega) \\ &= \sum_{k=-\infty}^{\infty} \sup_{2^k \leq \omega < 2^{k+1}} |\ln \omega|^{m-1} \frac{\sin^2(\omega/2)}{\omega^{2\alpha}} \\ &\leq C \left(\sum_{k=0}^{\infty} |k|^{m-1} 2^{-2k(1-\alpha)} + \sum_{k=0}^{\infty} |k|^{m-1} 2^{-2\alpha k} \right) < \infty. \end{aligned}$$

Also note that for $\alpha < s$, g is integrable and

$$
\begin{aligned}
G(\xi) &= \int_0^\infty g(\omega)\omega^{2(s-\alpha)-1+i\xi}d\omega = \int_0^\infty \sin^2(\omega/2)\omega^{i\xi-2\alpha-1}d\omega \\
&= -\frac{\sqrt{\pi}\Gamma(-\alpha+\frac{i\xi}{2})}{4^{(\alpha+1-\frac{i\xi}{2})}\Gamma(\alpha+\frac{1}{2}-\frac{i\xi}{2})} \neq 0
\end{aligned}
\tag{5.5}
$$

for all ξ (where $\Gamma(\xi)$ is the gamma function which has no zero and has simple poles at $\xi = 0, -1, -2, \ldots$; the calculation is by Mathematica). Hence the conditions of g in Theorem 5.9 are fulfilled. Together with Theorem 5.4, there exists a nonzero q such that $q(2T) = q(T)$ for all T, and

$$\lim_{T\to\infty}\left(\frac{1}{T^{2(s-\alpha)}(\ln T)^{m-1}}\int_{-T}^{T}|\omega^s\hat{f}(\omega)|^2 d\omega - q(T)\right) = 0.$$

6. Higher-order L^2-Lipschitz exponent. In this section, we consider the higher-order difference so that the Lipschitz exponent is allowed to be greater than 1. For any interger $l > 0$, we define the lth-order difference of a function $g \in L^2(\mathbb{R})$ by

$$\Delta_h^{(l)} f(x) = \sum_{k=0}^{l}(-1)^k \binom{l}{k} f(x-kh)$$

and the L^2-Lipschitz exponent of g by

$$\alpha := L^2\text{-Lip}(g) = \inf\left\{\beta > 0 :\ 0 < \limsup_{h\to 0^+}\frac{1}{h^{2\beta}}\int_{-\infty}^{\infty}|\Delta_h^{(l)} g(t)|^2\, dt\right\}. \tag{6.1}$$

It is well known that $0 \le \alpha \le l$. For $0 < \alpha < 1$, the definition used in (5.4) coincides with new definition here, but for $\alpha = 1$, the two definitions may or may not be the same. We will clarify this situation in the following. Furthermore, we show that the asymptotic properties in the last section are also preserved for higher-order cases.

For simplicity, we only consider the case $l = 2$. Let f be the L^2_c-solution of (1.1), let

$$\tilde{\Phi}_n(h) = \int_{-\infty}^{\infty}\Delta_h^{(2)} f(x+n)\Delta_h^{(2)} f(x)dx$$

and let $\tilde{\mathbf{\Phi}}(h) = [\tilde{\Phi}_0(h), \tilde{\Phi}_1(h), \ldots, \tilde{\Phi}_N(h)]$.

LEMMA 6.1. *Suppose f is the L^2_c-solution of* (1.1). *Then for any* $\mathbf{u} \in \mathbb{C}^{N+1}$,

$$\langle\tilde{\mathbf{\Phi}}(h), \mathbf{u}\rangle = C\left\langle \mathbf{\Phi}(h) - \frac{1}{4}\mathbf{\Phi}(2h), \mathbf{u}\right\rangle. \tag{6.2}$$

Proof. Let $\mathbf{u} \in \mathbb{C}^{N+1}$. Then

$$
\begin{aligned}
\langle\tilde{\mathbf{\Phi}}(h), \mathbf{u}\rangle &= \sum_{n=0}^{N} u_n \int_{-\infty}^{\infty}\Delta_h^{(2)} f(x+n)\Delta_h^{(2)} f(x)dx \\
&= C\sum_{n=0}^{N} u_n \int_{-\infty}^{\infty}|\hat{f}(\omega)|^2 e^{in\omega}\sin^4(h\omega)d\omega
\end{aligned}
$$

$$= C\sum_{n=0}^{N} u_n \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 e^{in\omega} \left(\sin^2(h\omega) - \frac{1}{4}\sin^2(2h\omega)\right) d\omega$$

$$= C\langle \mathbf{\Phi}(h) - \frac{1}{4}\mathbf{\Phi}(2h), \mathbf{u}\rangle.$$

LEMMA 6.2. *Let λ be an eigenvalue of $\mathbf{W}_N^+$, $\lambda \neq 0$ or 2.*

(i) *Let* $\mathbf{u}$ *be a λ-eigenvector. Then*

$$\langle \tilde{\mathbf{\Phi}}(h), \mathbf{u}\rangle \begin{cases} \not\equiv 0 & \text{if } \lambda \neq \frac{1}{2} \\ \equiv 0 & \text{if } \lambda = \frac{1}{2}. \end{cases}$$

(ii) *Suppose $\lambda = \frac{1}{2}$ has order $m > 1$, and let* $\mathbf{u}$ *be such that* $(\mathbf{W}_N^+ - \frac{1}{2}\mathbf{I})^{m-1}\mathbf{u} \neq 0$. *Then*

$$\langle \tilde{\mathbf{\Phi}}(h), \mathbf{u}\rangle = \tilde{p}(h)(\ln h)^{m-2}h^2 + \delta(h), \tag{6.3}$$

where $\tilde{p}$ is a nonzero bounded continuous function with $\tilde{p}(h) = \tilde{p}(2h)$ and $\delta(h)$ has order smaller than $(\ln h)^{m-2}h^2$.

Proof. (i) Note that for $\beta = -\ln(\lambda/2)/(2\ln 2)$, Lemma 5.1 implies that

$$\langle \mathbf{\Phi}(h), \mathbf{u}\rangle = p(h)h^{2\beta}$$

for some nonzero bounded continuous p such that $p(h) = p(2h)$. Hence by (6.2),

$$\langle \tilde{\mathbf{\Phi}}(h), \mathbf{u}\rangle = C\left(1 - \frac{2^{2\beta}}{4}\right)p(h)h^{2\beta}, \tag{6.4}$$

and the result follows.

(ii) Theorem 5.2 implies that $\langle \mathbf{\Phi}(h), \mathbf{u}\rangle$ has order $(\ln h)^{m-1}p(h)h^2$ as $h \to 0$, and $\frac{1}{4}\langle \mathbf{\Phi}(2h), \mathbf{u}\rangle$ has order $\frac{1}{4}(\ln 2h)^{m-1}p(2h)(2h)^2 = (\ln 2h)^{m-1}p(h)h^2$. Note that $(\ln h)^{m-1} - (\ln 2h)^{m-1}$ is of order $(\ln h)^{m-2}$. Consequently,

$$\langle \tilde{\mathbf{\Phi}}(h), \mathbf{u}\rangle = \left\langle \mathbf{\Phi}(h) - \frac{1}{4}\mathbf{\Phi}(2h), \mathbf{u}\right\rangle = \tilde{p}(h)(\ln h)^{m-2}h^2 + \delta(h),$$

where $\tilde{p}$ and $\delta(h)$ is as asserted (the order of $\delta(h)$ follows from the same argument and Theorem 5.2).

By using Lemma 6.2(i), we can extend Corollary 5.8 as follows.

PROPOSITION 6.3. *Suppose f is an L_c^2-solution of* (1.1). *Then the L^2-Lipschitz exponent of f is given by*

$$0 < \alpha = -\ln(\tilde{\Lambda}_{\max}/2)/(2\ln 2) \leq 2, \tag{6.5}$$

where

$$\tilde{\Lambda}_{\max} := \max\left\{|\lambda'| : \lambda' \textit{ eigenvalue of } \mathbf{W}_N^+ \textit{ and } |\lambda'| < \frac{1}{2}\right\}$$

if $\frac{1}{2}$ is an eigenvalue of order 1 *and is the only eigenvalue of modulus* $\frac{1}{2}$;

$$\tilde{\Lambda}_{\max} := \Lambda_{\max}$$

otherwise.

In the first case, f is differentiable a.e. and $f' \in L^2(\mathbb{R})$ (see the proof of Corollary 5.5), and f has Lipschitz exponent > 1. In the other case, the new and old definitions coincide.

The corresponding extension for Theorem 5.4 is the following.

THEOREM 6.4. *For the above α, let m be the highest order among those λ such that $|\lambda| = \tilde{\Lambda}_{\max}$. Then*

$$\lim_{h\to 0+}\left(\frac{1}{h^{2\alpha}|\ln h|^{m-1}}\int_{-\infty}^{\infty}|\Delta_h^{(2)}f(x)|^2dx - p(h)\right) = 0 \tag{6.6}$$

except for the special case where $\lambda = \frac{1}{2}$ and the order m is strictly greater then the other eigenvalues of moduli $\frac{1}{2}$; in such a case, $\alpha = 1$ and

$$\lim_{h\to 0+}\left(\frac{1}{h^{2}|\ln h|^{m-2}}\int_{-\infty}^{\infty}|\Delta_h^{(2)}f(x)|^2dx - p(h)\right) = 0. \tag{6.6$'$}$$

For the Fourier asymptotic result corresponding to Theorem 5.10, we note that

$$\begin{aligned}\tilde{\varphi}(h) &= \frac{1}{h^{2\alpha}|\ln h|^{m-1}}\int_{-\infty}^{\infty}|\Delta_h^{(2)}f(x)|^2dx \\ &= \frac{1}{h^{2\alpha}|\ln h|^{m-1}}\int_{-\infty}^{\infty}|\hat{f}(\omega)|^2\sin^4(\frac{h\omega}{2})d\omega.\end{aligned}$$

By taking

$$g(\omega) = \frac{\sin^4(\omega/2)}{|\omega|^{2s}}$$

and observing that for $\alpha < s$, g is integrable and

$$\begin{aligned}G(\xi) &= \int_0^{\infty} g(\omega)\omega^{2(s-\alpha)-1+i\xi}d\omega = \int_0^{\infty}\sin^4(\omega/2)\omega^{i\xi-2\alpha-1}d\omega \\ &= \frac{\sqrt{\pi}(4^{(\alpha-1-\frac{i\xi}{2})}-1)\Gamma(-\alpha+\frac{i\xi}{2})}{4^{(\alpha+1-\frac{i\xi}{2})}\Gamma(\alpha+\frac{1}{2}-\frac{i\xi}{2})} \neq 0\end{aligned}$$

for all ξ (compare this with (5.5)), we have the following.

THEOREM 6.5. *Under the same hypotheses as in Theorem 6.4, for $\alpha < s$, except for the special case,*

$$\lim_{T\to\infty}\left(\frac{1}{T^{2(s-\alpha)}|\ln T|^{m-1}}\int_{-T}^{T}|\omega^s\hat{f}(\omega)|^2d\omega - q(T)\right) = 0 \tag{6.7}$$

for some nonzero bounded continuous q (depending on s) such that $q(2T) = q(T)$. For the special case, we have

$$\lim_{T\to\infty}\left(\frac{1}{T^{2(s-1)}|\ln T|^{m-2}}\int_{-T}^{T}|\omega^s\hat{f}(\omega)|^2d\omega - q(T)\right) = 0. \tag{6.7$'$}$$

For the third-order difference $\Delta_h^{(3)} f$, we can use

$$\sin^6 x = \frac{15}{16}\sin^2 x - \frac{3}{8}\sin^2 2x + \frac{1}{16}\sin^2 3x$$

to replace the relationship in (6.2). For eigenvalues $\lambda = \frac{1}{2}$ or $\frac{1}{8}$ and the order m is as in the special case in Theorem 6.4 (the corresponding α are 1 and 2, respectively), the logarithmic terms in the asymptotic formulas are of order $m-2$. For the other cases, they are $m-1$. The higher-order difference $\Delta_h^{(l)} f$ behaves the same way.

As an example, we consider Daubechies's well-known scaling function D_4. Let f be the solution of (1.1) with coefficients

$$c_0 = \frac{1+\sqrt{3}}{4}, \quad c_3 = \frac{1-\sqrt{3}}{4}, \quad c_1 = 1 - c_3, \quad c_2 = 1 - c_0.$$

It follows from a direct calculation that

$$\mathbf{W}_3^+ = \begin{pmatrix} 2 & 0 & 0 & 0 \\ \frac{9}{4} & 1 & -\frac{1}{8} & 0 \\ 0 & 2 & 0 & 0 \\ -\frac{1}{4} & \frac{9}{8} & \frac{9}{8} & -\frac{1}{8} \end{pmatrix}$$

and the eigenvalues are 2, $\frac{1}{2}$, and $-\frac{1}{8}$, where $\frac{1}{2}$ has order 2. It fits into the above special case. By Theorem 5.4, $\alpha = 1$ and

$$\frac{1}{h^2|\ln h|}\int_{-\infty}^{\infty} |\Delta_h f(x)|^2 dx \approx p(h)$$

as $h \to 0^+$. It is known that f is differentiable a.e. [D], [DL2], but the asymptotic property implies that $f' \notin L^2(\mathbb{R})$. If we consider the second-order difference, then by a direct calculation and making use of the expressions in Theorem 5.2 and (6.4), we have

$$\frac{1}{h^2}\int_{-\infty}^{\infty} |\Delta_h^{(2)} f(x)|^2 dx \approx 2p(h)$$

as $h \to 0$.

For the Fourier transformation, we cannot apply Theorem 5.10 since $\alpha = 1$, but we can use (6.7)$'$ derived from the higher-order difference. It implies that for $1 < s$, there exists a bounded continuous q (depends on s) satisfying $q(T) = q(2T)$ and

$$\psi(T) = \frac{1}{T^{2(s-1)}}\int_{-T}^{T} |\omega^s \hat{f}(\omega)|^2 d\omega \approx q(T) \tag{6.8}$$

as $T \to \infty$. We include some graphic illustrations of this in the appendix.

Appendix. For the four-coefficient dilation equation

$$f(x) = c_0 f(2x) + c_1 f(2x-1) + c_2 f(2x-2) + c_3 f(2x-3)$$

with $c_0 + c_2 = 1$, $c_1 + c_3 = 1$, we use c_0 and c_3 as independent parameters to plot the various regions and functions.

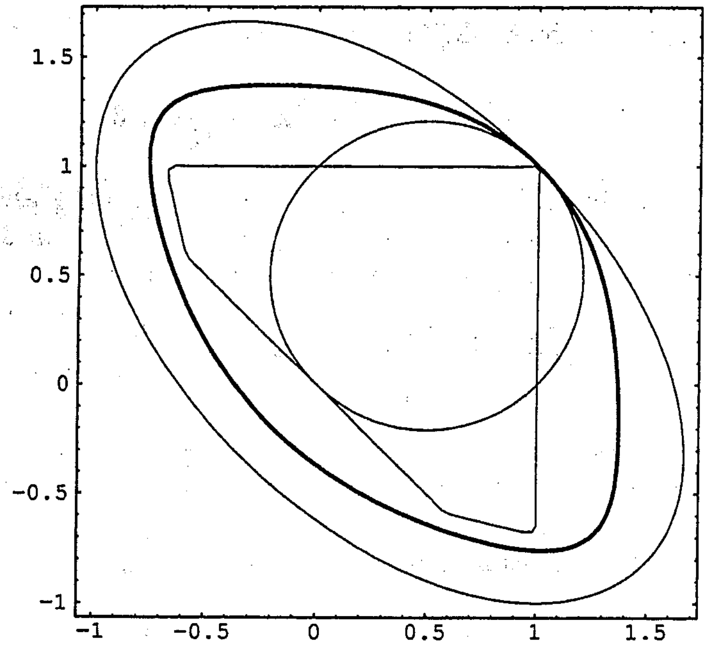


FIG. 1.

In Figure 1, the region bounded by the thicker curve corresponding to those (c_0, c_3) where all eigenvalues of $\mathbf{W}_3^+$ are less than or equal to 2 (2 is simple except at $(c_0, c_3) = (1, 1)$). This is the exact region where the L_c^2-solution exists [LW1].

The circular curve

$$\left(c_0 - \frac{1}{2}\right)^2 + \left(c_3 - \frac{1}{2}\right)^2 = \frac{1}{2}$$

is the circle of orthogonality [La]. The wavelet generated by the corresponding scaling function is orthonormal.

The triangular region is an approximation where the joint spectral radius of T_0 and T_1 restricted on H are less than 1 and the continuous solutions exist [DL1], [CH].

The ellipse is given by

$$c_0^2 + c_3^2 - c_0 - c_3 + c_0 c_3 = 0,$$

outside which no L^1-solution exists [H].

Figure 2(a) is the graph of the L^2-Lipschitz exponent $\alpha = -\ln(\Lambda_{\max}/2)/(2\ln 2)$.

Figure 2(b) is the graph of α on the circle of orthogonality, plotted in terms of the angles. Note that D_4 is the smoothest one on the circle.

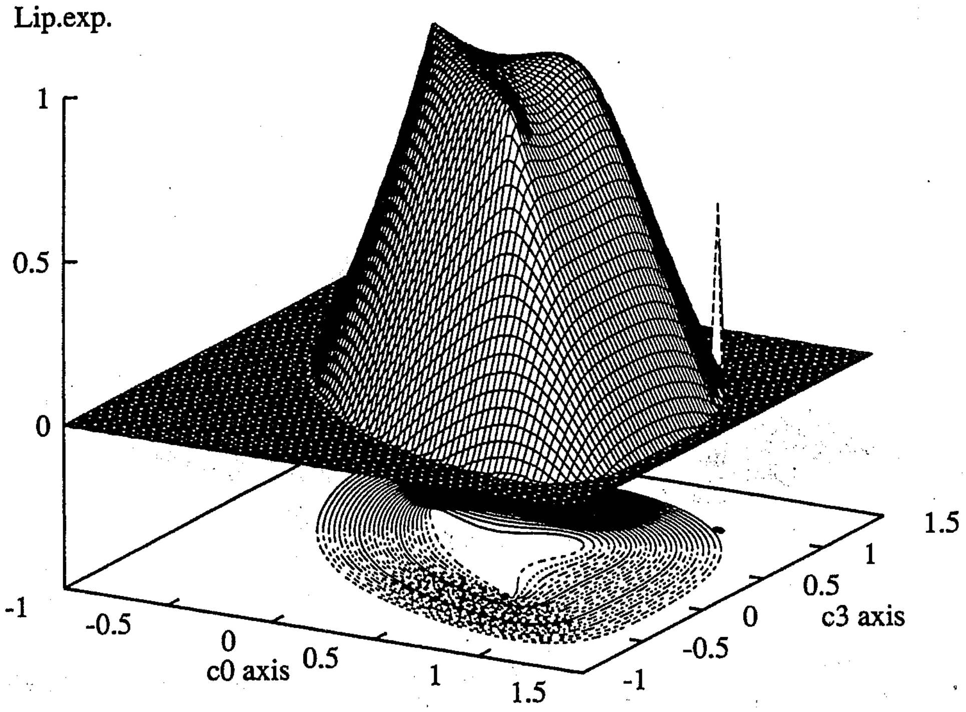
Figure 3(a) is the graph of the L^2-Lipschitz exponent $\alpha = -\ln(\tilde{\Lambda}_{\max}/2)/(2\ln 2)$, using the second-order difference.

Figure 3(b) is the cross-section of $c_0 + c_3 = 1$; Figure 3(c) is the cross-section of $c_0 = c_3$.

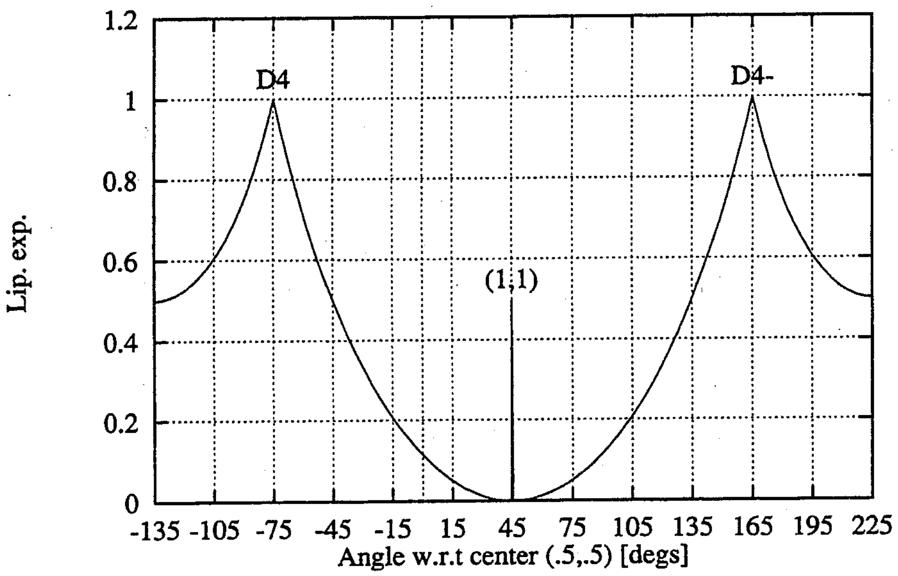
Figure 4(a) is the Daubechies scaling function $f = D_4$.

Figure 4(b) is its Fourier transformation $\hat{f}(\omega)$.

Figures 4(c), 4(e), and 4(g) are $\omega^s \hat{f}(\omega)$ with $s = 1.5, 1.25, 1.00$, respectively, and Figures 4(d) and 4(f) are the corresponding averages $\psi(T) = \frac{1}{T^{2(s-1)}} \int_{-T}^{T} |\omega^s \hat{f}(\omega)|^2 d\omega$.



(a)



(b)

FIG. 2.

Note that the $\psi(T)$ are approximately multiplicative periodic as $T \to \infty$. They follows from (6.5). We cannot draw such conclusions from the theorem for $s = \alpha = 1$ (see Figure 4(h)). However, if we take $\psi(T) = \frac{1}{\ln T} \int_{-T}^{T} |\omega \hat{f}(\omega)|^2 d\omega$, then it looks multiplicative periodic as in Figure 4(i); we have no proof for that yet.

Acknowledgments. The graphs in the appendix are due to Mr. Wonkoo Kim, to whom we express our deep gratitude.

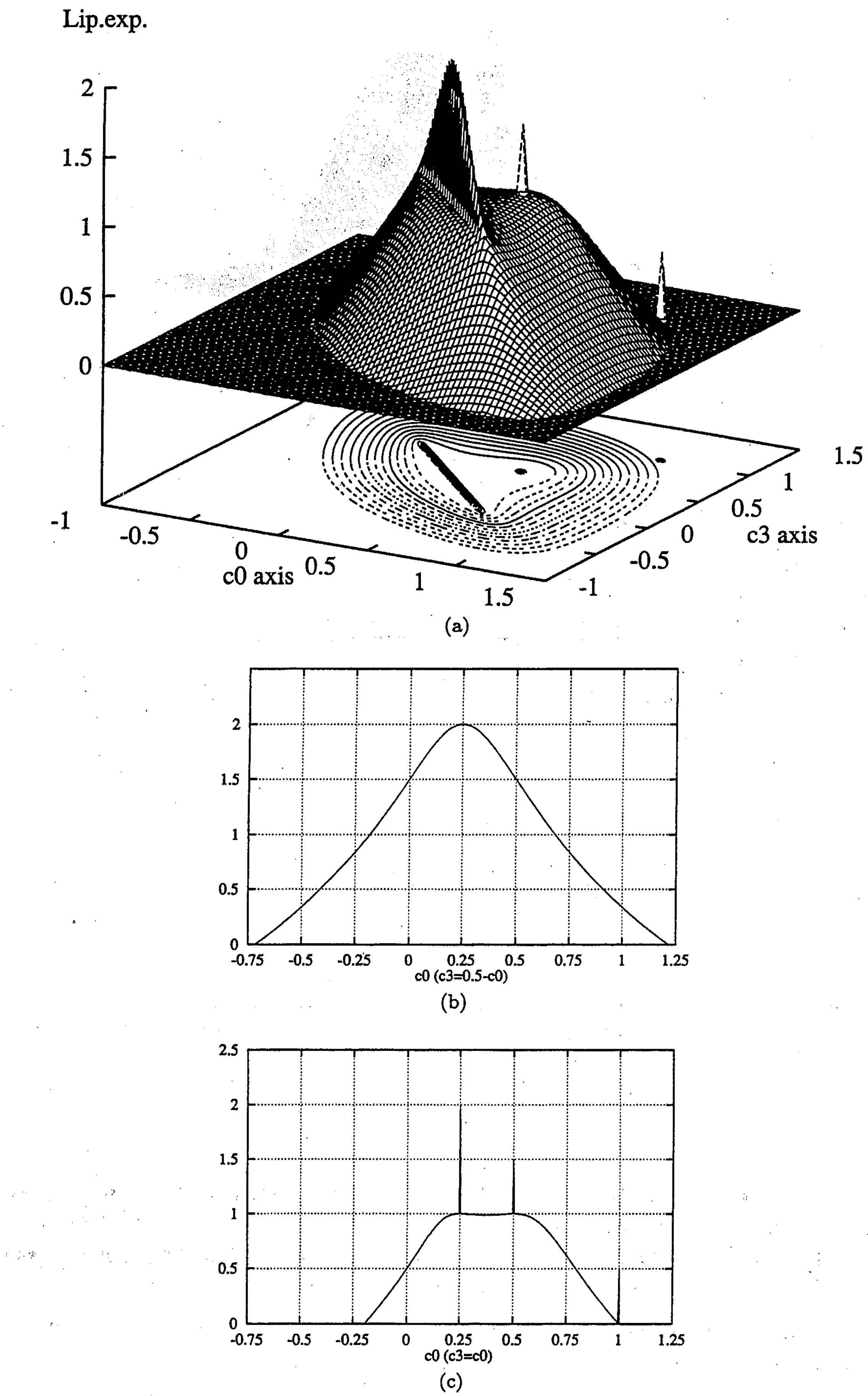


FIG. 3.

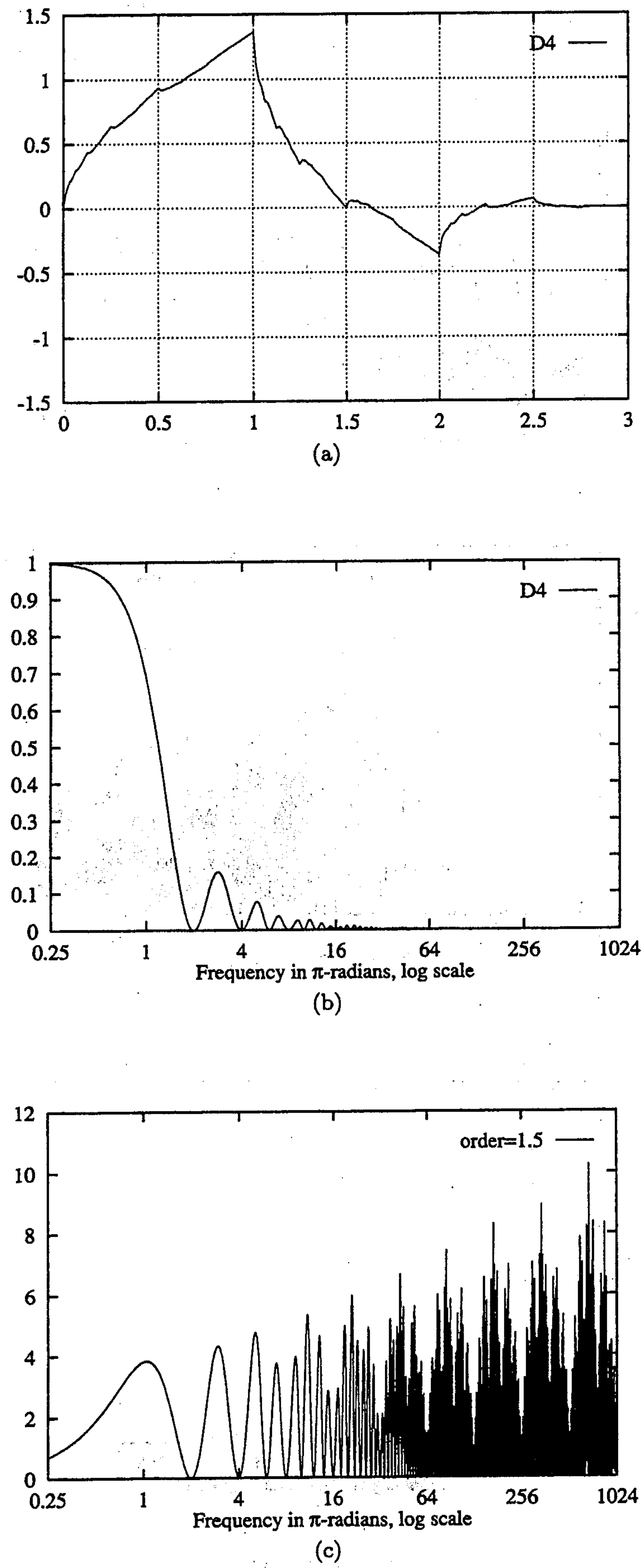
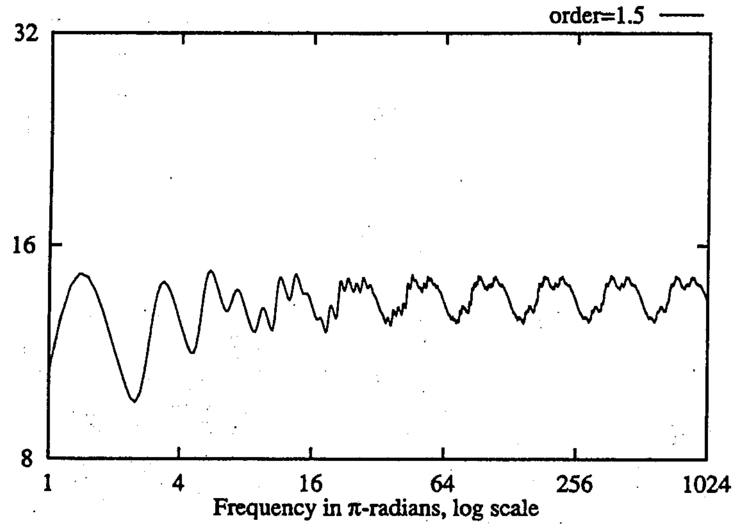
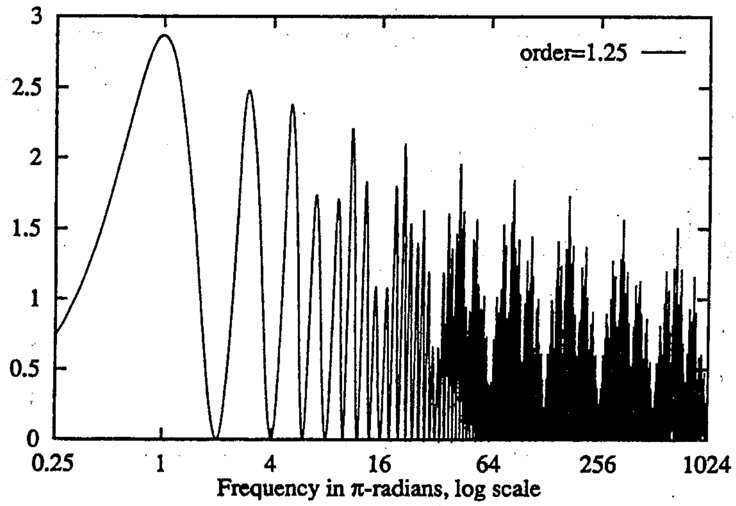

D4
1.5
1
0.5
0
-0.5
-1
-1.5
0 0.5 1 1.5 2 2.5 3
(a)
D4
1 0.9 0.8 0.7 0.6 0.5 0.4 0.3 0.2 0.1 0
0.25 1 4 16 64 256 1024
Frequency in π-radians, log scale
(b)
order=1.5
12 10 8 6 4 2 0
0.25 1 4 16 64 256 1024
Frequency in π-radians, log scale
(c)

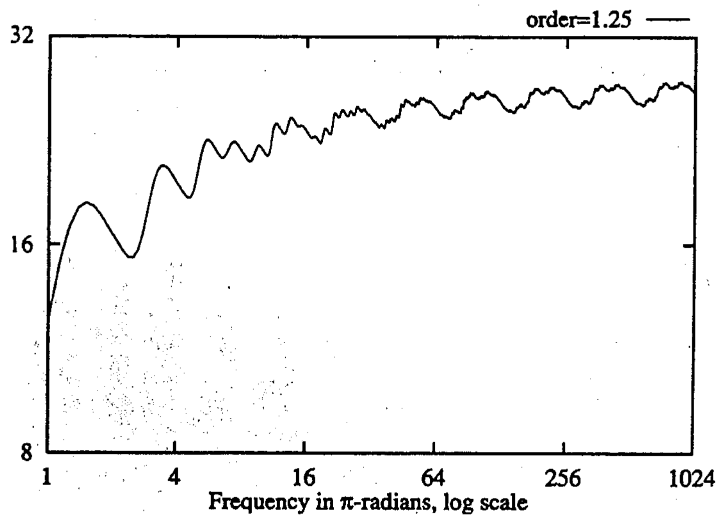
FIG. 4.



(d)

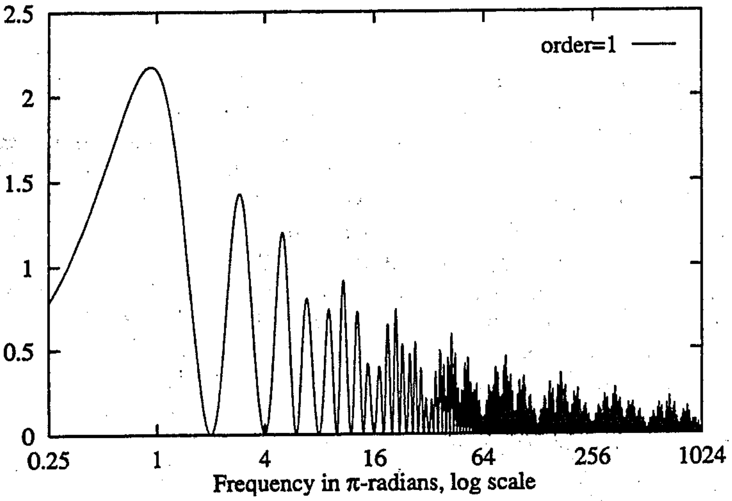


(e)

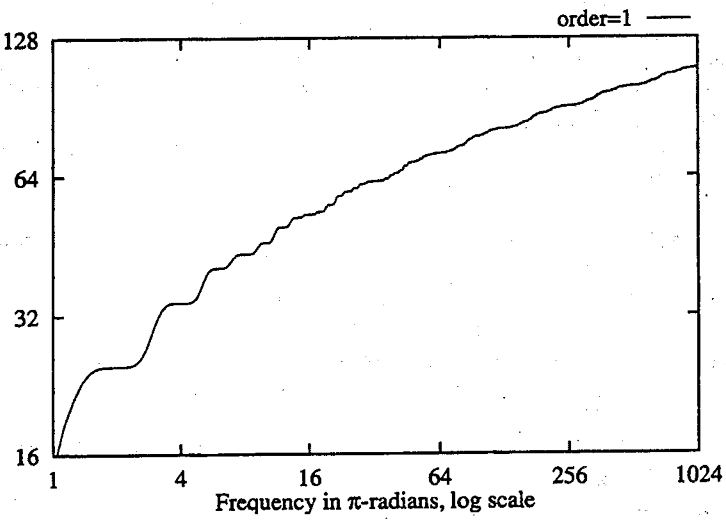


(f)

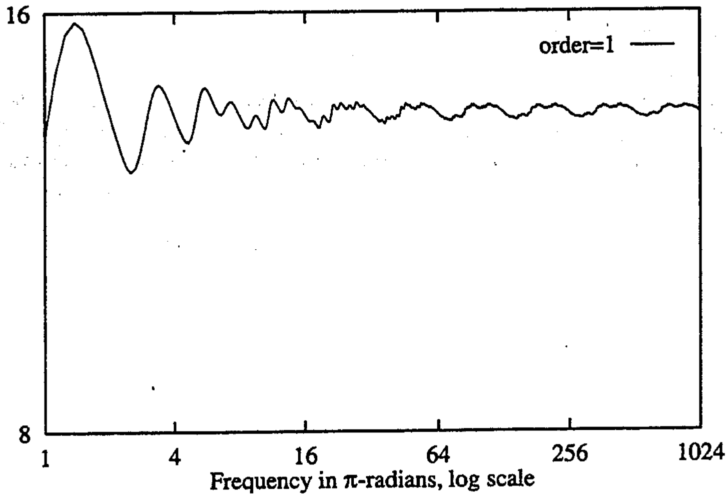
FIG. 4. (*cont.*)



(g)



(h)



(i)

FIG. 4. (*cont.*)

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